

Instabilities

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1 Two-stream instability

1.1 Dispersion relation

This instability occurs when the electrons and ions have a relative velocity. Working in the frame moving with the ions, we have:

Ions: $n = n_0 + n_i$; $\vec{v} = \vec{v}_i$; ($\vec{v}_0 \equiv 0$ for the ions.)

Electrons: $n = n_0 + n_e$; $\vec{v} = \vec{v}_0 + \vec{v}_e$

There is no magnetic field, $\vec{B}_0 = 0$, and we look for an electrostatic perturbation so $\vec{B}_1 = 0$ too. Then the equations of motion are:

$$\text{ions: } Mn_0 \frac{\partial \vec{v}_i}{\partial t} = en_0 \vec{E} \quad (1)$$

and

$$\text{electrons: } mn_0 \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_e \right] = -en_0 \vec{E} \quad (2)$$

Poisson's equation:

$$\vec{\nabla} \cdot \vec{E} = \frac{e(n_i - n_e)}{\epsilon_0} \quad (3)$$

Continuity:

$$\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot (n_0 \vec{v}_i) = 0 \quad (4)$$

for ions, and

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot ([n_0 + n_e] [\vec{v}_0 + \vec{v}_e]) = 0 \quad (5)$$

for electrons. Linearizing, and using the usual wave form for the perturbed quantities, we have:

Eqn (1)

$$-i\omega Mn_0 \vec{v}_i = en_0 \vec{E} \quad (6)$$

Eqn (2)

$$-imn_0 \vec{v}_e (\omega - \vec{k} \cdot \vec{v}_0) = -en_0 \vec{E} \quad (7)$$

Eqn (3)

$$i\vec{k} \cdot \vec{E} = \frac{e(n_i - n_e)}{\varepsilon_0} \quad (8)$$

Eqn (4)

$$-i\omega n_i + in_0 \vec{k} \cdot \vec{v}_i = 0 \quad (9)$$

and eqn (5)

$$-i\omega n_e + i\vec{k} \cdot (n_0 \vec{v}_e + n_e \vec{v}_0) = 0 \quad (10)$$

Solving for the densities, we have from (9):

$$n_i = n_0 \frac{\vec{k} \cdot \vec{v}_i}{\omega} \quad (11)$$

while from (10)

$$n_e = n_0 \frac{\vec{k} \cdot \vec{v}_e}{\omega - \vec{k} \cdot \vec{v}_0} \quad (12)$$

The denominator here indicates the doppler-shifted frequency in the electron rest frame.

Then from Poisson's equation (8):

$$i\vec{k} \cdot \vec{E} = \frac{en_0}{\varepsilon_0} \left(\frac{\vec{k} \cdot \vec{v}_i}{\omega} - \frac{\vec{k} \cdot \vec{v}_e}{\omega - \vec{k} \cdot \vec{v}_0} \right)$$

Finally we use the equations of motion (6) and (7):

$$i\vec{k} \cdot \vec{E} = \frac{en_0}{\varepsilon_0} \left(\frac{\vec{k} \cdot e\vec{E}}{-i\omega^2 M} - \frac{\vec{k} \cdot e\vec{E}}{im(\omega - \vec{k} \cdot \vec{v}_0)^2} \right)$$

Since we expect $\vec{k} \cdot \vec{E}$ to be non zero in an electrostatic wave, we divide it out to get the dispersion relation:

$$1 = \omega_{pe}^2 \left[\frac{m}{M\omega^2} + \frac{1}{(\omega - \vec{k} \cdot \vec{v}_0)^2} \right] \quad (13)$$

1.2 Growth rate

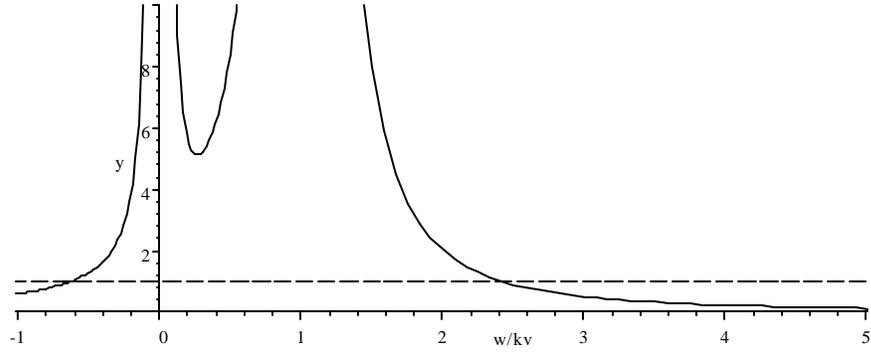
Let's try to find the growth rate for the 2-stream instability. The dispersion relation (13) is a fourth order equation for ω .

$$\omega^2 (\omega - \vec{k} \cdot \vec{v}_0)^2 = \omega^2 \omega_{pe}^2 + (\omega - \vec{k} \cdot \vec{v}_0)^2 \omega_{pi}^2$$

If $\vec{k} \cdot \vec{v}_0 = 0$, we get $\omega^2 = \omega_{pe}^2 + \omega_{pi}^2$, with no imaginary part to ω . These are just plasma oscillations. Thus the interesting case is propagation parallel to

the stream: $\vec{k} \cdot \vec{v}_0 = kv_0$. Calling the right hand side of equation (13) $F(\omega)$, we see that $F \rightarrow 0$ as $\omega \rightarrow \pm\infty$, and $F \rightarrow \infty$ as $\omega \rightarrow 0$ or kv_0 . Thus there is a minimum at point A where $0 < \omega < kv_0$. If this minimum is less than 1, then there will be 4 real roots, implying that the system is stable. But if the minimum is greater than 1, there will be only 2 real roots, and two complex roots (a complex conjugate pair). One of these complex roots corresponds to growth, and one to damping. Thus the condition for instability is:

$$F(\omega_A) > 1 \quad (14)$$



Plot shows $F(\omega)$ with $kv_0 = \omega_p/\sqrt{2}$ and $m/M = 0.1$

First we find the position of the minimum.

$$\frac{dF}{d\omega} = \omega_p^2 \left(-2 \frac{m/M}{\omega^3} - \frac{2}{(\omega - \vec{k} \cdot \vec{v}_0)^3} \right)$$

Set this equal to zero:

$$-2 \frac{m/M}{\omega^3} - \frac{2}{(\omega - \vec{k} \cdot \vec{v}_0)^3} = 0 \implies \frac{\omega}{(\omega - \vec{k} \cdot \vec{v}_0)} = \left(-\frac{m}{M} \right)^{1/3} = -\left(\frac{m}{M} \right)^{1/3}$$

where we have taken the real cube root. This gives:

$$\omega_A = kv_0 \left(\frac{m}{M} \right)^{1/3} \left(1 + \left(\frac{m}{M} \right)^{1/3} \right)^{-1} \quad (15)$$

and

$$\omega_A - kv_0 = kv_0 \left(\frac{\left(\frac{m}{M} \right)^{1/3}}{1 + \left(\frac{m}{M} \right)^{1/3}} - 1 \right) = \frac{kv_0}{1 + \left(\frac{m}{M} \right)^{1/3}}$$

Then

$$\begin{aligned}
F(\omega_A) &= \omega_p^2 \left(\frac{(m/M) \left(1 + \left(\frac{m}{M}\right)^{1/3}\right)^2}{(kv_0)^2 (m/M)^{2/3}} + \frac{\left(1 + \left(\frac{m}{M}\right)^{1/3}\right)^2}{(kv_0)^2} \right) \\
&= \frac{\omega_p^2}{(kv_0)^2} \left(1 + \left(\frac{m}{M}\right)^{1/3}\right)^2 \left(1 + \left(\frac{m}{M}\right)^{1/3}\right) \\
&= \frac{\omega_{pe}^2}{(kv_0)^2} \left(1 + \left(\frac{m}{M}\right)^{1/3}\right)^3
\end{aligned} \tag{16}$$

Thus instability should occur for:

$$kv_0 \leq \omega_{pe} \left(1 + (m/M)^{1/3}\right)^{3/2} \tag{17}$$

For a given v_0 , this relation gives the maximum k (minimum λ) for instability. Small enough k (long enough λ) are always unstable. Equivalently, for a given v_0 and a given k , it gives a minimum density (in ω_{pe}).

Another way to look at this is that electron plasma oscillations, Doppler shifted to the ion frame, resonate with the ion plasma oscillations.

Now rewrite the dispersion relation (13):

$$\frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{(\omega - \vec{k} \cdot \vec{v}_0)^2} = 1$$

Rearrange:

$$(\omega - \vec{k} \cdot \vec{v}_0)^2 = \omega_{pe}^2 \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right)^{-1}$$

We expect to find that the real part of the frequency is roughly comparable to $\omega_{pe} \gg \omega_{pi}$ (we can check this later) so we approximate:

$$(\omega - \vec{k} \cdot \vec{v}_0) = \pm \omega_{pe} \left(1 + \frac{\omega_{pi}^2}{2\omega^2}\right) \tag{18}$$

where the terms match up as follows:

$$\vec{k} \cdot \vec{v}_0 \simeq \pm \omega_{pe}$$

and

$$\omega_0^3 \simeq \pm \frac{\omega_{pe} \omega_{pi}^2}{2}$$

so

$$\omega_0 = \frac{\omega_{pe}}{2^{1/3}} \left(\frac{m}{M}\right)^{2/3}, \text{ and } \omega_0 = \frac{\omega_{pe}}{2^{1/3}} \left(\frac{m}{M}\right)^{2/3} e^{\pm i\pi/3}$$

The last two roots form a complex conjugate pair:

$$\omega_0 = \frac{\omega_{pe}}{2^{1/3}} \left(\frac{m}{M}\right)^{2/3} \frac{1}{2} (1 \pm i\sqrt{3})$$

and one of these roots represents growth at rate

$$\gamma = \frac{\omega_{pe}}{2^{1/3}} \left(\frac{m}{M}\right)^{2/3} \frac{\sqrt{3}}{2}$$

Check:

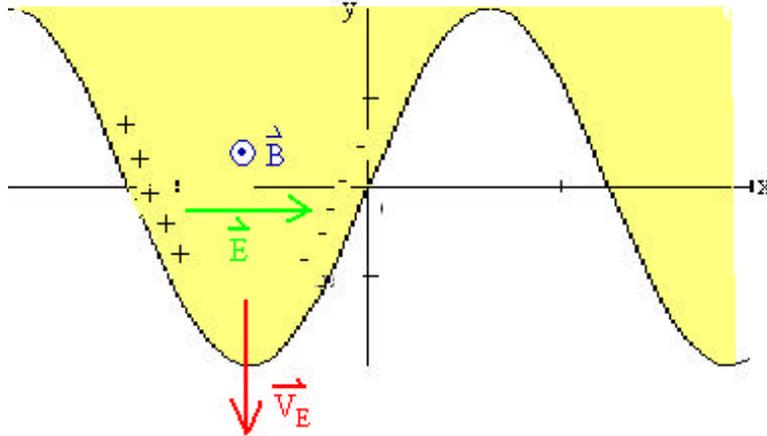
$$\frac{\omega_0}{\omega_{pi}} \simeq \left(\frac{\omega_{pe}}{\omega_{pi}}\right)^{1/3} = \left(\frac{M}{m}\right)^{1/3} \simeq 12 \gg 1$$

so our approximation is valid.

2 Rayleigh Taylor Instability

2.1 Drift Analysis

Consider a plasma supported against gravity by a magnetic field \vec{B} . The magnetic field acts like a "light fluid" of density $\sim \mu_0 B^2$ supporting the "heavy" plasma. The system is unstable to overturning, putting the heavy fluid on the bottom.



The gravitational drift (motion notes eqn 7) is

$$\vec{v}_{Dg} = \frac{m \vec{g} \times \vec{B}}{q B^2} = \frac{m (-g\hat{y}) \times B\hat{z}}{q B^2} = -\frac{m g \hat{x}}{q B} \quad (19)$$

So electrons go in the plus x -direction, and protons go in the $-x$ -direction. Thus we get a charge separation as shown in the figure, which, in turn, produces an electric field in the x -direction.

Now the $\vec{E} \times \vec{B}$ drift

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} \quad (20)$$

is the same for both protons and electrons. Since \vec{E} is in the x -direction, \vec{v}_E is in the direction of $\hat{x} \times \hat{z} = -\hat{y}$. Thus the perturbation continues to grow.

Let the perturbation be of the form

$$\Delta y = a \sin kx \quad (21)$$

where the amplitude a is initially small ($a \ll \lambda$). Since the gravitational drift speed is proportional to m , we neglect the electrons. The equation of charge continuity is:

$$\frac{\partial \rho_q}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \quad (22)$$

Thus at the boundary, the surface charge density σ grows because the current density brings charge to the surface. Integrate over a pillbox crossing the surface, as usual:

$$\int \frac{\partial \rho_q}{\partial t} dV = - \oint \hat{n} \cdot \vec{j} dA$$

Thus

$$\frac{\partial \sigma}{\partial t} = -\hat{n} \cdot \vec{j} \quad (23)$$

where \hat{n} is the inward (upward here) pointing normal. The surface has slope

$$\frac{\partial y}{\partial x} = ka \cos kx = \tan \theta$$

and thus a vector tangent to the surface has components:

$$\hat{t} = (\cos \theta, \sin \theta) = \left(\frac{1}{\sqrt{1 + (ka \cos kx)^2}}, \frac{ka \cos kx}{\sqrt{1 + (ka \cos kx)^2}} \right)$$

and thus the normal we want has components:

$$\hat{n} = \left(\frac{-ka \cos kx}{\sqrt{1 + (ka \cos kx)^2}}, \frac{1}{\sqrt{1 + (ka \cos kx)^2}} \right)$$

For small amplitude perturbations, $ka \ll 1$, we may approximate:

$$\hat{n} \approx (-ka \cos kx, 1) \quad (24)$$

Then, with \vec{v}_{Dg} in the x -direction,

$$\vec{j} = n_0 e \vec{v}_i = -n_0 M \frac{g}{B} \hat{x}$$

and equation (23) becomes

$$\frac{\partial \sigma}{\partial t} = -n_0 M \frac{g}{B} k a \cos kx = -n_0 M \frac{g}{B} k a \cos kx \quad (25)$$

So we may write

$$\sigma = \sigma_0 \cos kx$$

where

$$\frac{d\sigma_0}{dt} = -n_0 M \frac{g}{B} k a \quad (26)$$

Now recall the dielectric constant for low frequency motions in a magnetized plasma (Chen equation 3-28, fluids notes eqn 1)

$$\varepsilon = \left(1 + \frac{\mu_0 \rho c^2}{B^2}\right) \varepsilon_0 \approx \frac{\mu_0 \rho c^2}{B^2} \varepsilon_0 \gg \varepsilon_0$$

We use this with Poisson's equation:

$$\vec{\nabla} \cdot \varepsilon \vec{E} = \rho$$

Integrating across the boundary, we have:

$$\varepsilon \vec{E} \cdot \hat{n} - \varepsilon_0 \vec{E}_V \cdot \hat{n} \simeq \varepsilon \vec{E} \cdot \hat{n} = \sigma$$

where the field \vec{E}_V on the vacuum side is unimportant since $\varepsilon \gg \varepsilon_0$. Thus,

$$\vec{E} \cdot \hat{n} = \frac{\sigma_0 \cos kx}{\varepsilon} = E_x n_x + E_y n_y \approx E_y \quad (27)$$

With our usual assumed form for perturbations, $e^{i(\vec{k} \cdot \vec{x} - \omega t)}$, and uniform density away from the boundary, the electric potential satisfies Laplace's equation:

$$\nabla^2 \Phi = 0 \implies k^2 \Phi = 0 \implies k_x^2 + k_y^2 = 0 \implies k_y = \pm i k_x \quad (28)$$

Thus

$$E_y = \frac{\sigma_0 \cos kx}{\varepsilon} e^{-ky} \quad (29)$$

and

$$E_x = \frac{\sigma_0 \sin kx}{\varepsilon} e^{-ky} \quad (30)$$

(Check : $\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{\sigma_0}{\varepsilon} (k \cos kx e^{-ky} + (-k) \cos kx e^{-ky}) = 0$ as required)

Now the y -component of the $\vec{E} \times \vec{B}$ drift moves the boundary, so:

$$\frac{\partial}{\partial t} \Delta y = v_{Ey}(y=0) = -\frac{E_x}{B} = -\frac{\sigma_0 \sin kx}{\varepsilon B} = \frac{da}{dt} \sin kx \quad (31)$$

where we used equation (21) in the last step. Thus, differentiating wrt t and using equation 26, we get

$$\begin{aligned}\frac{d^2 a}{dt^2} &= -\frac{1}{\varepsilon B} \frac{d\sigma_0}{dt} = -\frac{1}{\varepsilon B} \left(-n_0 M \frac{g}{B} k a \right) = \frac{1}{\varepsilon B^2} n_0 M g k a \\ &= \frac{1}{\frac{\mu_0 \rho c^2}{B^2} \varepsilon_0 B^2} n_0 M g k a = \frac{1}{\varepsilon_0 \mu_0 \rho c^2} n_0 M g k a = g k a\end{aligned}\quad (32)$$

This is the exponential equation with solutions $e^{\pm\gamma t}$ and $\gamma = \sqrt{gk}$. The positive root indicates growth of the perturbation.

Note that any drift that has a similar effect to the gravitational drift (*e.g.* curvature drift) would cause a similar instability.

2.2 Normal mode analysis

We can get the same result by several different methods. Next we consider a somewhat more powerful method that allows to consider an equilibrium state that is non-uniform.

Consider a plasma at rest in the lab, supported by a magnetic field. We choose coordinates such that $\vec{g} = -g\hat{y}$ and $\vec{B}_0 = B_0(y)\hat{z}$. First we investigate the initial equilibrium. From the momentum equation:

$$\vec{\nabla} \left(P_0 + \frac{\mu_0}{2} B_0^2 \right) = \rho_0 \vec{g} + \mu_0 \left(\vec{B}_0 \cdot \vec{\nabla} \right) \vec{B}_0 = \rho_0 \vec{g} \quad (33)$$

since \vec{B}_0 is not a function of z in this case. We also have Maxwell's equation $\vec{\nabla} \cdot \vec{B} = 0$, which is automatically satisfied by our assumed form for \vec{B}_0 . Finally, we choose $\vec{v}_0 = 0$. Since \vec{g} is in the $-y$ direction, equation 33 may be written as:

$$\frac{\partial}{\partial y} \left(P_0 + \frac{\mu_0}{2} B_0^2 \right) = -\rho_0 g \quad (34)$$

Next we perturb and linearize. .

Momentum equation:

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\vec{\nabla} P_1 - \mu_0 \vec{\nabla} \left(\vec{B}_0 \cdot \vec{B}_1 \right) + \mu_0 \left(\vec{B}_0 \cdot \vec{\nabla} \right) \vec{B}_1 + \mu_0 \left(\vec{B}_1 \cdot \vec{\nabla} \right) \vec{B}_0 + \rho_1 \vec{g} \quad (35)$$

Maxwell equation:

$$\vec{\nabla} \cdot \vec{B}_1 = 0 \quad (36)$$

Continuity equation:

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{v}_1) = 0 \quad (37)$$

Equation of state:

$$\begin{aligned}\frac{d}{dt} \left(\frac{P}{\rho^\gamma} \right) &= 0 \\ \implies \frac{1}{\rho_0^\gamma} \frac{\partial}{\partial t} P_1 + p_0 \left(-\frac{\gamma}{\rho_0^{\gamma+1}} \frac{\partial \rho_1}{\partial t} \right) + (\vec{v}_1 \cdot \vec{\nabla}) \left(\frac{P}{\rho^\gamma} \right) &= 0\end{aligned}\quad (38)$$

Magnetic field evolution:

$$\frac{\partial \vec{B}_1}{\partial t} + (\vec{v}_1 \cdot \vec{\nabla}) \vec{B}_0 + \vec{B}_0 (\vec{\nabla} \cdot \vec{v}_1) - (\vec{B}_0 \cdot \vec{\nabla}) \vec{v}_1 = 0 \quad (39)$$

Now let's look at these equations component by component. Since the equilibrium state has gradients in the y -direction, then we assume all perturbations are of the form $s_1(y)e^{i(kx-\omega t)}$, with no z -dependence.

The momentum equation has three components:

$$-i\omega\rho_0v_x = -ik_xP_1 - ik_x\mu_0B_0B_z \quad (40)$$

$$-i\omega\rho_0v_y = -\frac{\partial P_1}{\partial y} - \mu_0\frac{\partial}{\partial y}(B_0B_z) - \rho_1g \quad (41)$$

$$-i\omega\rho_0v_z = \mu_0B_y\frac{\partial B_0}{\partial y} \quad (42)$$

Maxwell equation for \vec{B}_1 :

$$ik_xB_x + \frac{\partial B_y}{\partial y} = 0 \quad (43)$$

Continuity equation:

$$-i\omega\rho_1 + \rho_0\vec{\nabla} \cdot \vec{v}_1 + v_y\frac{\partial \rho_0}{\partial y} = 0 \quad (44)$$

Equation of state:

$$-\frac{i\omega}{\rho_0^\gamma}P_1 + \frac{i\omega\gamma}{\rho_0^{\gamma+1}}P_0\rho_1 + v_y\frac{\partial}{\partial y}\left(\frac{P_0}{\rho_0^\gamma}\right) = 0 \quad (45)$$

Expand out the last term, and multiply by ρ_0^γ :

$$-i\omega P_1 + i\omega\gamma\frac{P_0}{\rho_0}\rho_1 + v_y\frac{\partial P_0}{\partial y} - \gamma v_y\frac{P_0}{\rho_0}\frac{\partial \rho_0}{\partial y} = 0 \quad (46)$$

Magnetic field evolution (3 components)

$$-i\omega B_x = 0 \quad (47)$$

$$-i\omega B_y = 0 \quad (48)$$

$$-i\omega B_z + v_y\frac{\partial B_0}{\partial y} + (\vec{\nabla} \cdot \vec{v}_1)B_0 = 0 \quad (49)$$

So $B_x = B_y = 0$. Thus the field lines do not bend. Now we solve for ρ_1 , P_1 , and B_z . From (44)

$$\rho_1 = \frac{1}{i\omega} \left(\rho_0\vec{\nabla} \cdot \vec{v}_1 + v_y\frac{\partial \rho_0}{\partial y} \right) \quad (50)$$

From (46) and (50)

$$\begin{aligned}
P_1 &= \gamma \frac{P_0}{\rho_0} \frac{1}{i\omega} \left(\rho_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial \rho_0}{\partial y} \right) + \frac{1}{i\omega} \left(v_y \frac{\partial P_0}{\partial y} - \gamma v_y \frac{P_0}{\rho_0} \frac{\partial \rho_0}{\partial y} \right) \\
&= \frac{1}{i\omega} \left(\gamma p_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial P_0}{\partial y} \right)
\end{aligned} \tag{51}$$

From (49)

$$B_z = \frac{1}{i\omega} \left[v_y \frac{\partial B_0}{\partial y} + (\vec{\nabla} \cdot \vec{v}_1) B_0 \right] = 0 \tag{52}$$

Notice that these three expressions have very similar forms. Now we put these expressions into the momentum equation components and solve for the components of \vec{v} .

$$\begin{aligned}
v_x &= \frac{1}{\omega \rho_0} \left(\frac{k_x}{i\omega} \right) \left(\gamma p_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial p_0}{\partial y} + \mu_0 B_0 \left(v_y \frac{\partial B_0}{\partial y} + (\vec{\nabla} \cdot \vec{v}_1) B_0 \right) \right) \\
&= -\frac{i}{\omega^2 \rho_0} k_x \left((\gamma p_0 + \mu_0 B_0^2) \vec{\nabla} \cdot \vec{v}_1 + v_y \left(\frac{\partial p_0}{\partial y} + \mu_0 B_0 \frac{\partial B_0}{\partial y} \right) \right)
\end{aligned} \tag{53}$$

From the initial equilibrium equation (34), we may simplify the factor multiplying v_y :

$$v_x = -\frac{i}{\omega^2 \rho_0} k_x \left((\gamma p_0 + \mu_0 B_0^2) \vec{\nabla} \cdot \vec{v}_1 - \rho_0 g v_y \right) \tag{54}$$

y component:

$$\begin{aligned}
i\omega \rho_0 v_y &= \frac{\partial}{\partial y} \frac{1}{i\omega} \left(\gamma p_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial p_0}{\partial y} \right) + \mu_0 \frac{\partial}{\partial y} \left[\frac{B_0}{i\omega} \left(v_y \frac{\partial B_0}{\partial y} + (\vec{\nabla} \cdot \vec{v}_1) B_0 \right) \right] \\
&\quad + \frac{1}{i\omega} \left(\rho_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial \rho_0}{\partial y} \right) g \\
v_y &= \frac{-1}{\omega^2 \rho_0} \left[\frac{\partial}{\partial y} \left((\gamma p_0 + \mu_0 B_0^2) \vec{\nabla} \cdot \vec{v}_1 + v_y \left(\frac{\partial p_0}{\partial y} + \mu_0 B_0 \frac{\partial B_0}{\partial y} \right) \right) \right] \\
&\quad - \frac{1}{\omega^2 \rho_0} \left(\rho_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial \rho_0}{\partial y} \right) g \\
&= \frac{-1}{\omega^2 \rho_0} \left[\frac{\partial}{\partial y} \left((\gamma p_0 + \mu_0 B_0^2) \vec{\nabla} \cdot \vec{v}_1 - \rho_0 g v_y \right) + \left(\rho_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial \rho_0}{\partial y} \right) g \right]
\end{aligned}$$

The z -component is trivial:

$$v_z = 0 \tag{56}$$

Now we use the equation (54) for v_x to eliminate the quantity $(\gamma p_0 + \mu_0 B_0^2) \vec{\nabla} \cdot \vec{v}_1$

$$i \frac{\omega^2 \rho_0}{k_x} v_x + \rho_0 g v_y = (\gamma p_0 + \mu_0 B_0^2) \vec{\nabla} \cdot \vec{v}_1 \tag{57}$$

Then v_y becomes:

$$\begin{aligned}
v_y &= \frac{-1}{\omega^2 \rho_0} \left[\frac{\partial}{\partial y} \left(i \frac{\omega^2 \rho_0}{k_x} v_x + \rho_0 g v_y - \rho_0 g v_y \right) + \left(\rho_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial \rho_0}{\partial y} \right) g \right] \\
&= \frac{-1}{\omega^2 \rho_0} \left[i \frac{\omega^2}{k_x} \frac{\partial}{\partial y} (\rho_0 v_x) + \left(\rho_0 \vec{\nabla} \cdot \vec{v}_1 + v_y \frac{\partial \rho_0}{\partial y} \right) g \right] \tag{58}
\end{aligned}$$

At this point we can simplify by considering only incompressible modes: $\vec{\nabla} \cdot \vec{v}_1 = 0$. We couldn't do this before, because incompressible is equivalent to $\gamma \rightarrow \infty$, and our equations contained a term $\gamma \vec{\nabla} \cdot \vec{v}_1$, which is therefore indeterminate. Now that we have eliminated this term, we are free to make the incompressible assumption. Note this does not mean the entire plasma is incompressible, but only that the modes we are investigating are incompressible.

Then:

$$\vec{\nabla} \cdot \vec{v}_1 = i k_x v_x + \frac{\partial v_y}{\partial y} = 0 \tag{59}$$

and the equation (58) for v_y becomes:

$$v_y = \frac{-i}{\rho_0 k_x} \frac{\partial}{\partial y} (\rho_0 v_x) - \frac{v_y}{\omega^2 \rho_0} \frac{\partial \rho_0}{\partial y} g \tag{60}$$

$$\begin{aligned}
v_y \left(1 + \frac{g}{\omega^2 \rho_0} \frac{\partial \rho_0}{\partial y} \right) &= \frac{-i}{\rho_0 k_x} \frac{\partial}{\partial y} \left(-\frac{\rho_0}{i k_x} \frac{\partial v_y}{\partial y} \right) = \frac{1}{\rho_0 k_x^2} \frac{\partial}{\partial y} \left(\rho_0 \frac{\partial v_y}{\partial y} \right) \\
&= \frac{1}{\rho_0 k_x^2} \frac{\partial \rho_0}{\partial y} \frac{\partial v_y}{\partial y} + \frac{1}{k_x^2} \frac{\partial^2 v_y}{\partial y^2} \tag{61}
\end{aligned}$$

Some possible solutions of this equation are:

1. Exponential density variations:

$$\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial y} = \text{constant} = \alpha \tag{62}$$

The equation for v_y becomes:

$$\frac{\partial^2 v_y}{\partial y^2} + \alpha \frac{\partial v_y}{\partial y} - k_x^2 v_y \left(1 + \frac{g}{\omega^2} \alpha \right) = 0 \tag{63}$$

This differential equation has a solution of the form $v_y = v_1 e^{py}$ where p is a solution of the equation:

$$p^2 + \alpha p - k_x^2 \left(1 + \frac{g}{\omega^2} \alpha \right) = 0 \tag{64}$$

The solution is $p = \pm \frac{1}{2} \sqrt{\left(\alpha^2 + 4k_x^2 + 4\frac{k_x^2}{\omega^2} g \alpha \right)} - \frac{1}{2} \alpha = \frac{\alpha}{2} \left(\pm \sqrt{1 + 4\frac{k_x^2}{\alpha^2} + 4\frac{k_x^2}{\omega^2} \frac{g}{\alpha}} - 1 \right)$,

and we expect $p \leq 0$ (the velocity dies away as we go away from the boundary) we need the minus sign if $\alpha > 0$. Then we see that $|p| > \alpha$. Thus

$$\frac{g\alpha}{\omega^2} = \frac{p^2 + \alpha p - k_x^2}{k_x^2} \Rightarrow \omega^2 = \frac{g\alpha k_x^2}{p(p + \alpha) - k_x^2}$$

We have $\omega^2 < 0$, and thus instability, if $k_x^2 > p(p + \alpha)$, and then with $\omega = i\gamma$, the growth rate γ is

$$\gamma = \sqrt{g\alpha} \frac{1}{\sqrt{1 - \frac{p(p+\alpha)}{k_x^2}}}$$

If $\omega^2 + g\alpha = 0$, there is a solution in which v_y does not vary with y ($p = 0$). Then we have an instability ($\omega = i\gamma$) with growth rate

$$\gamma = \sqrt{g\alpha}$$

. Note we have instability only if the density is increasing upward. If the density decreases upward (α is negative) then the plasma is stable.

2. No gradients in the initial state: $\frac{\partial \rho_0}{\partial y} = 0$. Instead we have a sharp boundary across which ρ_0 goes to zero, and the magnetic field suddenly increases. (This is the case that we analyzed using drifts above.) Then for $y > 0$ we have from equation (61):

$$v_y = \frac{1}{k_x^2} \frac{\partial^2 v_y}{\partial y^2} \implies v_y = v_1 e^{\pm k_x y} \quad (65)$$

For $y > 0$, the minus sign in the exponent is the appropriate choice. At the boundary itself, $\frac{\partial \rho_0}{\partial y} \rightarrow \infty$, and equation (61) becomes:

$$v_y \frac{g}{\omega^2 \rho_0} \frac{\partial \rho_0}{\partial y} = \frac{1}{\rho_0 k_x^2} \frac{\partial \rho_0}{\partial y} \frac{\partial v_y}{\partial y} \quad (66)$$

or

$$v_y \frac{g}{\omega^2 \rho_0} = \frac{1}{\rho_0 k_x^2} (-k_x v_y) \implies \omega^2 = -g k_x \quad (67)$$

Again we have an instability with growth rate $\gamma = \sqrt{g k_x}$, the same result we got from the drift analysis.

Note this is a pure instability (not overstability) since the frequency has no real part. The growth rate increases as \sqrt{k} , so smaller wavelengths grow faster. At some point this trend is limited by effects we have ignored in this analysis, such as viscosity, which is more important on small scales. Also, once the perturbations become non-linear, this growth rate no longer applies.

2.3 Instability of magnetically-confined plasma

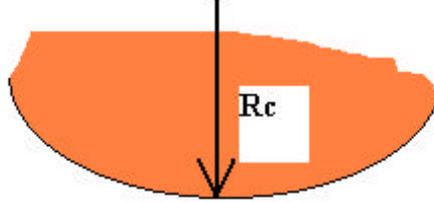
The curvature plus grad- \vec{B} drift is:

$$\vec{v}_{\text{total drift}} = \frac{m}{qB^2} \left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2 \right) \frac{\vec{R}_c}{R_c^2} \times \vec{B}$$

Comparing with the gravitational drift, we have an effective gravitational acceleration of

$$\vec{g}_{\text{eff}} = \frac{\vec{R}_c}{R_c^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right)$$

and thus we have an instability if \vec{R}_c points out of the plasma, as shown below:



Unstable situation

2.3.1 Energy analysis

The stable state is one of minimum energy. The energy of this system is the magnetic field energy ($u_B = B^2/2\mu_0$) and internal (thermal) energy. For an adiabatic equation of state, $P \propto \rho^\gamma$, the internal energy density is:

$$u_{\text{int}} = \frac{\mathcal{N}}{2} nkT$$

where the number of degrees of freedom \mathcal{N} is related to γ by

$$\begin{aligned} \gamma &= \frac{\mathcal{N}+2}{\mathcal{N}} \\ \mathcal{N} &= \frac{2}{\gamma-1} \end{aligned}$$

Thus

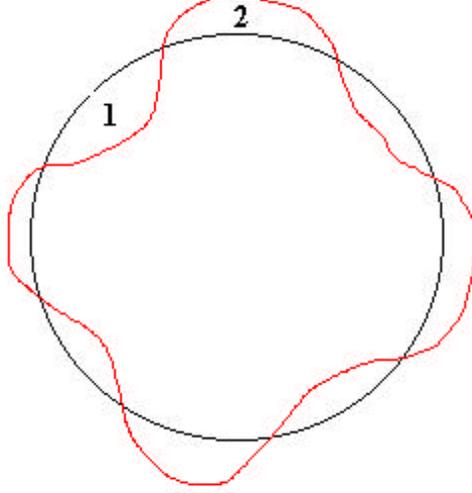
$$u_{\text{int}} = \frac{nkT}{\gamma-1} = \frac{P}{\gamma-1}$$

and the internal energy in volume V is

$$U_{\text{int}} = \frac{PV}{\gamma-1}$$

In the Rayleigh-Taylor instability driven by gravity, flux tubes are interchanged without any change in volume. There is no change in magnetic or internal energy. The gravitational energy is decreased as the plasma falls. The system is unstable.

Now we look at the magnetically confined plasma without gravity. Seen end on, the system looks like this:



flutes- end view

The system is low β , the plasma pressure is small compared with the magnetic pressure, so the magnetic energy density does not change much as the flux-tube 1 moves to position 2. The magnetic energy in a flux tube is:

$$E_M = \int_{\text{tube}} \frac{B^2}{2\mu_0} dV = \int_{\text{tube}} \frac{B^2}{2\mu_0} A dl$$

where A is the cross-sectional area of the tube and we integrate along its length. Now the flux $\Phi = BA$ is constant along the length of the tube, so

$$E_M = \frac{\Phi^2}{2\mu_0} \int_{\text{tube}} \frac{dl}{A} \quad (68)$$

Thus the change in energy due to the interchange of the flux tubes is:

$$\Delta E_M = \frac{\Phi_1^2}{2\mu_0} \int_2 \frac{dl}{A} + \frac{\Phi_2^2}{2\mu_0} \int_1 \frac{dl}{A} - \frac{\Phi_1^2}{2\mu_0} \int_1 \frac{dl}{A} - \frac{\Phi_2^2}{2\mu_0} \int_2 \frac{dl}{A}$$

and this change is zero if the two fluxes are equal: $\Phi_1 = \Phi_2$.

Now we look at the internal energy. As the flux tubes are interchanged, the pressure will change if the volume changes. The new pressure in region 2 is

$$P'_2 = P_1 \left(\frac{V_1}{V_2} \right)^\gamma$$

Thus the change in energy is

$$\begin{aligned}\Delta E_{\text{int}} &= \frac{1}{\gamma-1} \left[P_1 \left(\frac{V_1}{V_2} \right)^\gamma V_2 - P_1 V_1 + P_2 \left(\frac{V_2}{V_1} \right)^\gamma V_1 - P_2 V_2 \right] \\ &= \frac{1}{\gamma-1} \left[P_1 V_1^\gamma V_2^{1-\gamma} - P_1 V_1 + P_2 V_2^\gamma V_1^{1-\gamma} - P_2 V_2 \right]\end{aligned}$$

Now letting $P_2 = P_1 + \delta P$, $V_2 = V_1 + \delta V$, and expanding to second order, we have:

$$\begin{aligned}\Delta E_{\text{int}} &= \frac{P_1 V_1}{\gamma-1} \left\{ \frac{1 + (1-\gamma) \frac{\delta V}{V} + \frac{(1-\gamma)(-\gamma)}{2} \left(\frac{\delta V}{V} \right)^2 - 1 +}{\left(1 + \frac{\delta P}{P} \right) \left[1 + \gamma \frac{\delta V}{V} + \frac{\gamma(\gamma-1)}{2} \left(\frac{\delta V}{V} \right)^2 \right]} - \left(1 + \frac{\delta P}{P} \right) \left(1 + \frac{\delta V}{V} \right) \right\} \\ &= \frac{P_1 V_1}{\gamma-1} \left[\frac{(1-\gamma) \frac{\delta V}{V} - \frac{\gamma(1-\gamma)}{2} \left(\frac{\delta V}{V} \right)^2 + 1}{+\gamma \frac{\delta V}{V} + \frac{\gamma(\gamma-1)}{2} \left(\frac{\delta V}{V} \right)^2 + \frac{\delta P}{P} + \gamma \frac{\delta P \delta V}{P V} - 1 - \frac{\delta P}{P} - \frac{\delta V}{V} - \frac{\delta P}{P} \frac{\delta V}{V}} \right] \\ &= \frac{P_1 V_1}{\gamma-1} \left[(\gamma-1) \frac{\delta P \delta V}{P V} + \gamma(\gamma-1) \left(\frac{\delta V}{V} \right)^2 \right] \\ &= P_1 V_1 \frac{\delta V}{V} \left[\frac{\delta P}{P} + \gamma \left(\frac{\delta V}{V} \right) \right] \\ &= \delta V \left(\delta P + \gamma \frac{P}{V} \delta V \right)\end{aligned}$$

The first order terms all disappear, as expected if the initial state is an equilibrium. Any equilibrium is an extremum of the energy, but a stable equilibrium is a minimum. Thus the energy change must be positive for the system to be stable. As $P \rightarrow 0$, the first term dominates. Thus for stability we need

$$\delta P > 0 \quad \text{and} \quad \delta V > 0$$

or

$$\delta P < 0 \quad \text{and} \quad \delta V < 0$$

Now with $\Phi_1 = \Phi_2$,

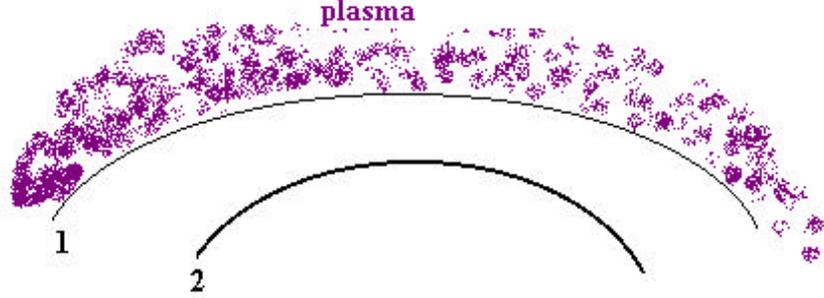
$$\begin{aligned}\delta V &= \delta \int A d\ell = \delta \int B A \frac{d\ell}{B} = \Phi \delta \int \frac{d\ell}{B} \\ &= \Phi \left[\int_2 \frac{d\ell}{B} - \int_1 \frac{d\ell}{B} \right]\end{aligned}$$

Thus if 1 represents the region with the plasma, so that $\delta P = P_2 - P_1 < 0$, we need $\delta V < 0$ for stability. That is

$$\int_2 \frac{d\ell}{B} < \int_1 \frac{d\ell}{B} \quad \text{for stability.}$$

But since $B_2 > B_1$ to confine the plasma, we need

$$l_2 < \frac{B_2}{B_1} l_1$$



stable system - side view

Thus the field lines curve as shown, with the plasma on the "outside". This is the geometry of a neutron star magnetosphere, but is the opposite of what we want for a CNF machine, where the plasma is on the "inside". This is a very dangerous instability, as the growth rate is high and it moves a lot of plasma!

3 Resistive Drift Waves

3.1 Two-fluid analysis

This is an example of a universal instability. We start off by analyzing drift waves with no resistivity, then add the resistivity and look at its effects.

Drift waves are driven by gradients in the initial state. Confined plasmas *always* have gradients, so these kinds of instabilities occur in many situations. As an example, we consider electrostatic drift waves, which have a phase velocity equal to the diamagnetic drift velocity:

$$\vec{v}_D = -\frac{k_B T}{q} \frac{\vec{\nabla} n \times \vec{B}}{n B^2} = \frac{k_B T}{m \omega_c} \frac{\vec{\nabla} n \times \vec{B}}{n B} \quad (69)$$

We use the 2-fluid equations and study small perturbations.

$$nm \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = nq \left(\vec{E} + \vec{v} \times \vec{B} \right) - \vec{\nabla} p \quad (70)$$

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n \vec{v}) = 0 \quad (71)$$

Now we perturb and linearize. In the perturbed state there is an electric field:

$$\vec{E} = -\vec{\nabla} \Phi$$

Electrons can move freely along the field lines, and so establish the Boltzmann distribution:

$$n = n_o \exp(e\Phi/k_B T) \Rightarrow n_1 = n_o \frac{e\Phi}{k_B T} \quad (72)$$

(We could also get this from the equation of motion.) Now let $\vec{B} = B_o \hat{z}$, and let $\vec{\nabla}n$ be along \hat{x} . Also take \vec{k} in the $y - z$ plane, $T_i = 0$, $\vec{v}_{i,o} = 0$ and $\vec{v}_{e,o} = \vec{v}_D = V_D \hat{y}$, with $V_D = -\frac{k_B T}{m \omega_c} \frac{1}{n} \frac{dn_o}{dx}$ (eqn.69).

Continuity equation (71)

$$-i\omega n_1 + i\vec{k} \cdot \vec{v}_i n_o + v_{ix} \frac{dn_o}{dx} = 0$$

Solve for the ion density n_1 :

$$\frac{n_1}{n_o} = \frac{v_{ix}}{i\omega n_o} \frac{dn_o}{dx} + \frac{\vec{k} \cdot \vec{v}_i}{\omega} \quad (73)$$

Ion equation of motion (eqn 70 for the ions)

$$-i\omega n M \vec{v}_i = en \left(-i\vec{k}\Phi \right) + en \vec{v}_i \times \vec{B}_o$$

Solve the equation of motion for the components of \vec{v} . (Remember that \vec{k} is in the $y - z$ plane.)

$$\begin{aligned} -i\omega M v_{ix} &= e v_{iy} B_o \Rightarrow v_{ix} = i \frac{c}{\omega} v_{iy} \\ -i\omega M v_{iy} &= -i k_y e \Phi - e v_{ix} B_o \\ -i\omega M v_{iz} &= -i k_z e \Phi \Rightarrow v_{iz} = \frac{e \Phi k_z}{\omega M} \end{aligned} \quad (74)$$

Substitute v_{ix} into equation for v_{iy} :

$$-i\omega M v_{iy} = -i k_y e \Phi - e i \frac{c}{\omega} v_{iy} B_o \Rightarrow v_{iy} = \frac{k_y e \Phi}{\omega M} \left(1 - \frac{c^2}{\omega^2} \right)^{-1} \quad (75)$$

Then we can find v_{ix} from eqn (74).

$$v_{ix} = i \frac{c}{\omega} \frac{k_y e \Phi}{\omega M} \left(1 - \frac{c^2}{\omega^2} \right)^{-1}$$

Now we specialize to low frequency waves with $\omega \ll c$, so that $1 - \frac{c^2}{\omega^2} \approx -\frac{c^2}{\omega^2}$. Then:

$$v_{ix} = -i \frac{c}{\omega} \frac{k_y e \Phi}{\omega M} \frac{\omega^2}{c} = -i k_y \frac{e \Phi}{c M}$$

and

$$v_{iy} = -\frac{k_y e \Phi}{\omega M} \frac{\omega^2}{c} = -\frac{k_y e \Phi}{M} \frac{\omega}{c}$$

and $|v_{iy}| \ll |v_{ix}|$. Put all this into eqn. 73 for n_1 , ignoring the term in v_{iy} compared with v_{ix} :

$$\begin{aligned} \frac{n_1}{n_o} &= \frac{1}{i\omega n_o} \left(-i k_y \frac{e \Phi}{c M} \right) \frac{dn_o}{dx} + \frac{k_z}{\omega} \frac{e \Phi k_z}{\omega M} \\ &= \frac{e \Phi}{k_B T} \left(\frac{k_y V_D}{\omega} + \frac{v_s^2 k_z^2}{\omega^2} \right) \end{aligned}$$

where $v_s^2 = \frac{k_B T}{M}$. Now we use the plasma approximation. Set this n_1 equal to the value in eqn. 72:

$$\frac{e\Phi}{k_B T} = \frac{e\Phi}{k_B T} \left(\frac{k_y V_D}{\omega} + \frac{v_s^2 k_z^2}{\omega^2} \right) \Rightarrow \omega^2 - \omega k_y V_D - v_s^2 k_z^2 = 0 \quad (76)$$

which is the dispersion relation for the drift waves. Notice that if $k_y \rightarrow 0$ we get ion acoustic waves, and if $k_z \rightarrow 0$ we get $\omega = k_y V_D$, waves with phase speed equal to the diamagnetic drift. These are the drift waves.

3.2 MHD analysis

We can analyze the same system with single fluid MHD. We do not have to use the plasma approximation, as it is "built-in", but now we include a non-zero current, carried by the electrons, in the initial state.

The single-fluid MHD equations are:

Momentum equation:

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = \vec{j} \times \vec{B} - \vec{\nabla} P \quad (77)$$

Ohm's law:

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} + \frac{1}{en} \left(\vec{j} \times \vec{B} - \vec{\nabla} P_e \right) \quad (78)$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad (79)$$

Note that in the initial equilibrium we have:

$$\vec{j}_o \times \vec{B}_o = \vec{\nabla} p \Rightarrow \vec{j}_o = -\frac{\vec{\nabla} p \times \vec{B}_o}{B_o^2} = -n_o e \vec{v}_D$$

i.e. the current is due to the electron's diamagnetic drift.

Now perturb and linearize. Because n_0 is not uniform, we assume $n_1 \propto n_0$, all other quantities are independent of x and proportional to $\exp[i(k_z z + k_y y)]$. For now, we neglect resistivity. Ohm's law gives the velocity:

$$\vec{E} = -\vec{\nabla} \Phi, \quad \text{so } \vec{E} = -i \vec{k} \Phi$$

y -component of Ohm's law:

$$-ik_y \Phi - v_x B_o \approx 0 \Rightarrow v_x = -ik_y \frac{\Phi}{B_o} \quad (80)$$

where we neglected the higher order terms.

x -component of Ohm's law:

$$v_y B_o \approx 0 \quad (81)$$

and z -component

$$-ik_z\Phi = \frac{1}{en_o} (-ik_z n_1 k_B T_e) \Rightarrow n_1 = n_o \frac{e\Phi}{k_B T_e} \quad (82)$$

which is just the Boltzmann relation. To get v_z we use the equation of motion. The x - and y -components give the current.

x -component

$$-i\omega\rho v_x = j_y B_o - n_1 k_B T_e \frac{1}{n_o} \frac{dn_o}{dx} \Rightarrow j_y = \frac{1}{B_o} \left(-i\omega\rho v_x + n_1 k_B T_e \frac{1}{n_o} \frac{dn_o}{dx} \right) \quad (83)$$

y -component

$$-i\omega\rho v_y = -j_x B_o - ik_y n_1 k_B T_e \Rightarrow j_x = \frac{1}{B_o} (i\omega\rho v_y - ik_y n_1 k_B T_e) = -ik_y \frac{n_1}{B_o} k_B T_e \quad (84)$$

z -component

$$-i\omega\rho v_z = -ik_z n_1 k_B T_e \Rightarrow v_z = \frac{k_z n_1}{\omega n_o} \frac{k_B T_e}{M} \quad (85)$$

The continuity equation gives:

$$-i\omega\rho_1 + \frac{d}{dx} (\rho_o v_x) + ik_y \rho_o v_y + ik_z \rho_o v_z = 0$$

Substituting for the velocity components (80), (81) and (85), we have:

$$-i\omega \frac{n_1}{n_o} + \frac{1}{n_o} \frac{dn_o}{dx} \left(-ik_y \frac{\Phi}{B_o} \right) + ik_z \frac{k_z n_1}{\omega n_o} \frac{k_B T_e}{M} = 0 \quad (86)$$

Now substitute in from eqn.82:

$$-i\omega \frac{e\Phi}{k_B T_e} - ik_y \frac{1}{n_o} \frac{dn_o}{dx} \frac{e\Phi}{k_B T_e} \left(\frac{k_B T_e}{eB_o} \right) + i \frac{k_z^2}{\omega} \frac{e\Phi}{k_B T_e} \frac{k_B T_e}{M} = 0$$

or

$$\omega^2 - \omega k_y V_D - v_s^2 k_z^2 = 0$$

as before. (Remember that V_D is negative.)

3.3 The instability

Inclusion of resistivity causes the electron density to deviate slightly from the Boltzmann relation. The z -component of Ohm's law becomes:

$$-ik_z\Phi = \eta j_z + \frac{1}{en_o} (-ik_z n_1 k_B T_e) \Rightarrow -ik_z\Phi \left(1 - \frac{n_1 k_B T_e}{n_o e\Phi} \right) = \eta j_z$$

Resistivity also produces a second-order correction to j_y and j_x , which we could get from the equation of motion, but we don't need those here. . Instead, we use the equation of charge continuity plus the plasma approximation to get:

$$\vec{\nabla} \cdot \vec{j} = 0 = \frac{\partial j_x}{\partial x} + ik_y j_y + ik_z j_z$$

Then using equations 83 and 84

$$\begin{aligned} \frac{\partial}{\partial x} \left(-ik_y \frac{n_1}{B_o} k_B T_e \right) + ik_y \frac{1}{B_o} \left(-i\omega \rho v_x + n_1 k_B T_e \frac{1}{n_o} \frac{dn_o}{dx} \right) + ik_z \left[-ik_z \frac{\Phi}{\eta} \left(1 - \frac{n_1}{n_o} \frac{k_B T_e}{e\Phi} \right) \right] = 0 \\ -ik_y \frac{n_1}{n_o} \frac{dn_o}{dx} \frac{k_B T_e}{B_o} + \omega n_o \frac{M}{B_o} k_y v_x + ik_y \frac{k_B T_e}{B_o} \frac{1}{n_o} \frac{dn_o}{dx} + k_z^2 \frac{\Phi}{\eta} \left(1 - \frac{n_1}{n_o} \frac{k_B T_e}{e\Phi} \right) = 0 \end{aligned}$$

The terms in k_y cancel. Then using equation 80 we have:

$$\omega n_o \frac{M}{B_o} k_y \left(-ik_y \frac{\Phi}{B_o} \right) + k_z^2 \frac{\Phi}{\eta} \left(1 - \frac{n_1}{n_o} \frac{k_B T_e}{e\Phi} \right) = 0$$

which gives the density:

$$\frac{n_1}{n_o} = \frac{e\Phi}{k_B T_e} \left[1 - i\eta \omega \left(\frac{k_y}{k_z} \right)^2 \frac{M n_o}{B_o^2} \right] \quad (87)$$

–the Boltzmann relation with a correction proportional to η . As expected, η causes the electron density to lag (notice the i indicating a phase shift of $\pi/2$).

Now we use the continuity equation (equation 86) again, eliminating n_1 .

$$\begin{aligned} \frac{e\Phi}{k_B T_e} \left[1 - i\eta \omega \left(\frac{k_y}{k_z} \right)^2 \frac{M n_o}{B_o^2} \right] \left(-i\omega + i \frac{k_z^2 k_B T_e}{\omega M} \right) - ik_y \frac{e\Phi}{k_B T_e} \frac{1}{n_o} \frac{dn_o}{dx} \left(\frac{k_B T_e}{e B_o} \right) = 0 \\ \left[1 - i\eta \omega \left(\frac{k_y}{k_z} \right)^2 \frac{M n_o}{B_o^2} \right] (\omega^2 - k_z^2 v_s^2) - k_y \omega V_D = 0 \end{aligned} \quad (88)$$

which is the new dispersion relation. Check that we get back eqn (13) as $\eta \rightarrow 0$.

We look for a solution of the form $\omega = \omega_r + i\gamma$ and expect to find $\gamma \ll \omega_r$ if η is small. Then

$$\omega^2 \approx \omega_r^2 + 2i\omega_r \gamma$$

and the dispersion relation becomes:

$$(\omega_r^2 + 2i\omega_r \gamma - k_z^2 v_s^2) \left[1 - i\eta (\omega_r + i\gamma) \left(\frac{k_y}{k_z} \right)^2 \frac{M n_o}{B_o^2} \right] - k_y (\omega_r + i\gamma) V_D = 0$$

Expanding, and keeping only first order terms, we have:

$$\begin{aligned} (\omega_r^2 + 2i\omega_r \gamma - k_z^2 v_s^2) \left[1 - i\eta \omega_r \left(\frac{k_y}{k_z} \right)^2 \frac{M n_o}{B_o^2} \right] - k_y (\omega_r + i\gamma) V_D = 0 \\ (\omega_r^2 - k_z^2 v_s^2) \left[1 - i\eta \omega_r \left(\frac{k_y}{k_z} \right)^2 \frac{M n_o}{B_o^2} \right] + 2i\omega_r \gamma - k_y (\omega_r + i\gamma) V_D = 0 \end{aligned}$$

Real part:

$$(\omega_r^2 - k_z^2 v_s^2) - k_y \omega_r V_D = 0$$

which is the same dispersion relation (76) that we obtained with $\eta = 0$.

Imaginary part:

$$2\omega_r \gamma - (\omega_r^2 - k_z^2 v_s^2) \eta \omega_r \left(\frac{k_y}{k_z}\right)^2 \frac{M n_o}{B_o^2} - k_y \gamma V_D = 0$$

Solve for γ :

$$\gamma = \frac{(\omega_r^2 - k_z^2 v_s^2) \eta \omega_r \left(\frac{k_y}{k_z}\right)^2 \frac{M n_o}{B_o^2}}{2\omega_r - k_y V_D} = \frac{k_y V_D \eta \omega_r^2 \left(\frac{k_y}{k_z}\right)^2 \frac{M n_o}{B_o^2}}{2\omega_r - k_y V_D}$$

where we used the dispersion relation for ω_r to simplify the expression in the numerator. Then if $k_z \ll k_y$, $\omega_r \approx k_y V_D$, the denominator $2\omega_r - k_y V_D \approx k_y V_D$, and we find:

$$\gamma \simeq \eta \omega_r^2 \left(\frac{k_y}{k_z}\right)^2 \frac{M n_o}{B_o^2} = \eta (k_y V_D)^2 \left(\frac{k_y}{k_z}\right)^2 \frac{M n_o}{B_o^2}$$

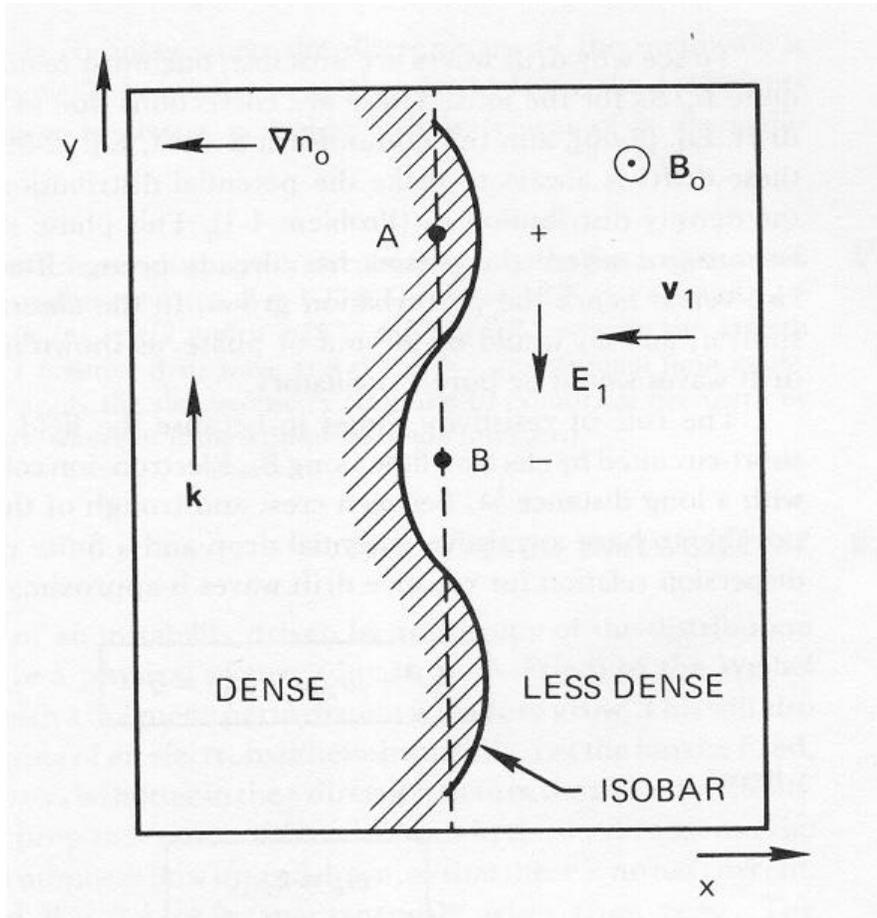
We can see that this equals Chen's expression in equation 6-66 by noting that:

$$\eta = \frac{m \nu_{ei}}{e^2 n_o}$$

Then:

$$\gamma = (k_y V_D)^2 \left(\frac{k_y}{k_z}\right)^2 \frac{m \nu_{ei}}{e^2 n_o} \frac{M n_o}{B_o^2} = (k_y V_D)^2 \left(\frac{k_y}{k_z}\right)^2 \frac{\nu_{ei}}{c \omega_c}$$

as given by Chen.



The diagram shows what is going on in this wave. Here \vec{k} is in the y -direction. Without resistivity, Φ is in phase with n_1 and $\vec{E} = -i\vec{k}\Phi$ is $\pi/2$ out of phase. \vec{E} is maximum where n_1 is zero, and has the direction, shown. As the wave propagates, the $\vec{E} \times \vec{B}$ drift serves to move the particles so as to maintain the wave. When there is a non-zero resistivity, the potential, and hence the field, gets further out of phase and the drift moves particles that are already displaced left further left, and the amplitude grows.

Here we have seen several different techniques for analyzing stability. We'll see some more when we tackle Vlasov theory.