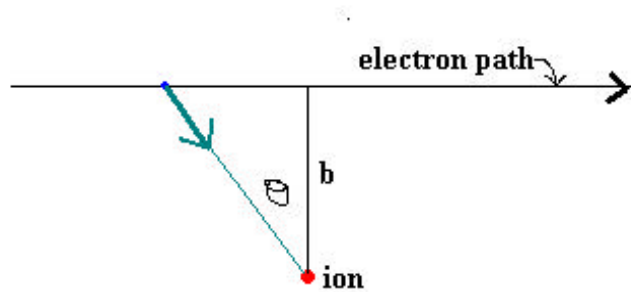# Bremmstrahlung

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An electron approaches a nucleus of charge Z. As a result of the encounter, the electron is accelerated and moves off in a different direction. In the absence of radiation, the electron would emerge from the encounter at the same speed that it entered. But the radiated photon decreases the electron's KE.

We start by looking at encounters in which the electron's direction of motion changes very little. The electric force produces an impulse

$$I = e \int E_\perp dt$$

perpendicular to the electron's path. In this approximation the impulse along the path is zero. The electric field component we need is

$$E_\perp = \frac{Ze}{r^2} \cos\theta = \frac{Zeb}{\left(b^2 + (vt)^2\right)^{3/2}}$$

Thus

$$I = Ze^2 b \int_{-\infty}^{+\infty} \frac{1}{\left(b^2 + (vt)^2\right)^{3/2}} dt$$

Let $vt = b \tan \theta$. Then

$$\begin{aligned} I &= Ze^2 b \int_{-\pi/2}^{\pi/2} \frac{(b/v) \sec^2 \theta d\theta}{b^3 \sec^3 \theta} \\ &= \frac{Ze^2}{bv} 2 \end{aligned}$$

Thus the electron's momentum changes by an amount

$$\Delta p = I$$

The length of the encounter is approximately $2b/v$, since the electric field decreases rapidly once $vt \geq b$. (This approximation gets better if the electron moves relativistically.) Thus

$$\frac{\Delta p}{\Delta t} \simeq \frac{Ze^2}{bv} 2 \frac{v}{2b} = \frac{Ze^2}{b^2}$$

and the average acceleration is

$$a = \frac{1}{m} \frac{\Delta p}{\Delta t} = \frac{Ze^2}{mb^2}$$

and it occurs over the interval $-\Delta t/2 < t < \Delta t/2$. Thus the Fourier transform is

$$a(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-b/v}^{+b/v} \frac{Ze^2}{mb^2} e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{Ze^2}{mb^2} \frac{e^{i\omega t}}{i\omega} \Big|_{-b/v}^{+b/v} = \frac{1}{\sqrt{2\pi}} \frac{2Ze^2}{mb^2} \frac{\sin \frac{\omega b}{v}}{\omega}$$

Thus, from the Larmor formula, the radiation spectrum is:

$$\frac{dP}{d\omega} \Big|_{\text{one electron}} = \frac{2}{3} \frac{e^2}{c^3} a(\omega)^2 = \frac{2}{3} \frac{e^2}{c^3} \left(\frac{1}{\sqrt{2\pi}} \frac{2Ze^2}{mb^2} \frac{\sin \frac{\omega b}{v}}{\omega} \right)^2$$

At low frequencies, $\omega \ll b/v$, the spectrum is flat:

$$\frac{dP}{d\omega} \Big|_{\text{one electron}} \simeq \frac{2}{3} \frac{e^2}{c^3} \left(\frac{1}{\sqrt{2\pi}} \frac{2Ze^2}{mbv} \right)^2 = \frac{4}{3\pi} \frac{Z^2 e^6}{m^2 b^2 v^2 c^3}$$

At higher frequencies, $\omega \gg v/b$, the emission oscillates wildly and is close to zero. Converting to frequency ν, where $\omega = 2\pi\nu$,

$$\frac{dP}{d\nu} = \frac{8}{3} \frac{Z^2 e^6}{m^2 b^2 v^2 c^3} \quad \text{for } \nu < \frac{2\pi v}{b}$$

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Now we sum up over all the electrons. The total number of electrons passing the ion between distances b and $b + db$ per second is $nv2\pi b\,db$ and thus

$$j_\nu = \frac{1}{4\pi} \int \frac{8}{3} \frac{Z^2 e^6}{m^2 b^2 v^2 c^3} nv2\pi b\,db = \int \frac{4}{3} \frac{Z^2 e^6}{m^2 bvc^3} n\,db = \frac{4}{3} \frac{Z^2 e^6}{m^2 vc^3} n \ln b$$

The factor $1/4\pi$ appears because j_ν is the emission per steradian.

Now we need to ponder the limits of the integral. The logarithm diverges at $b = 0$ and at $b = \infty$, so there must be maximum and minimum values of b. We already found $b_{\max}$ above: it is approximately $2\pi v/\nu$. Quantum mechanical considerations determine $b_{\min}$.

$$j_\nu = \frac{4}{3} \frac{Z^2 e^6}{m^2 vc^3} n \ln \frac{b_{\max}}{b_{\min}}$$

Now this result still assumes only one ion interacting with all the electrons. Of course, there are many ions. So accounting for interactions with all the ions, we have:

$$j_\nu = \frac{4}{3} \frac{Z^2 e^6}{m^2 vc^3} n_e n_i \ln \frac{b_{\max}}{b_{\min}}$$

The last step is to integrate over the electron speed distribution- a Maxwellian in a thermal plasma. The lower limit is not zero, because the electron cannot radiate more than its own energy. Thus

$$\frac{1}{2} m v_{\min}^2 = h\nu$$

Then:

$$
\begin{aligned}
j_\nu &= \frac{4}{3} \frac{Z^2 e^6}{m^2 c^3} n_e n_i \ln \frac{b_{\max}}{b_{\min}} \int_{\sqrt{2h\nu/m}}^{\infty} \frac{1}{v} \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) 4\pi v^2\,dv \\
&= \frac{16\pi}{3} \frac{Z^2 e^6}{m^2 c^3} \left(\frac{m}{2\pi kT}\right)^{3/2} n_e n_i \ln \frac{b_{\max}}{b_{\min}} \frac{kT}{m} \int_{\sqrt{2h\nu/m}}^{\infty} e^{-\frac{1}{2}m\frac{v^2}{kT}}\,d\left(\frac{mv^2}{2kT}\right) \\
&= \frac{16\pi}{3(2\pi)^{3/2}} \frac{Z^2 e^6}{m^2 c^3} \left(\frac{m}{kT}\right)^{1/2} n_e n_i \ln \frac{b_{\max}}{b_{\min}} \left. -e^{-\frac{1}{2}m\frac{v^2}{kT}} \right|_{\sqrt{2h\nu/m}}^{\infty} \\
&= \frac{8}{3\sqrt{2\pi}} \frac{Z^2 e^6}{m^2 c} \left(\frac{m}{kT}\right)^{1/2} n_e n_i \ln \frac{b_{\max}}{b_{\min}} e^{-\frac{h\nu}{kT}}
\end{aligned}
$$

This result is an approximation. Compare the exact result

$$j_\nu = \frac{16}{3} \left(\frac{\pi}{6}\right)^{1/2} \frac{Z^2 e^6}{m^2 c} \left(\frac{m}{kT}\right)^{1/2} n_e n_i e^{-\frac{h\nu}{kT}} g$$

where g is the Gaunt factor. We have captured the correct dependence on all the physical parameters. The ratio of the two expressions is

$$\frac{\text{approx}}{\text{exact}} = \frac{8}{3\sqrt{2\pi}}\frac{3}{16}\sqrt{\frac{6}{\pi}} = 0.28$$

It is also useful to look at the total energy radiated, as this leads to cooling of the plasma. The electrons radiate, but collisions reestablish equilibrium

$$
\begin{aligned}
P &= 4\pi \int j_\nu d\nu = 4\pi \frac{16}{3}\left(\frac{\pi}{6}\right)^{1/2}\frac{Z^2 e^6}{m^2 c}\left(\frac{m}{kT}\right)^{1/2} n_e n_i \int_0^\infty e^{-\frac{h\nu}{kT}} g \, d\nu \\
&= 4\pi \frac{16}{3}\left(\frac{\pi}{6}\right)^{1/2}\frac{Z^2 e^6}{m^2 c}\left(\frac{m}{kT}\right)^{1/2} n_e n_i \overline{g}\frac{kT}{h} \\
&= 4\pi \frac{16}{3}\left(\frac{\pi}{6}\right)^{1/2}\frac{Z^2 e^6}{m^{3/2} c}\frac{\sqrt{kT}}{h} n_e n_i \overline{g}
\end{aligned}
$$

The important features to notice are that $P \propto n^2 T^{1/2}$.

Putting in numbers, we get

$$j_\nu = 5.4 \times 10^{-39} Z^2 \frac{n_e n_i}{T^{1/2}} e^{-h\nu/kT} g_{ff} \ \text{erg} \cdot \text{cm}^{-3} \cdot \text{s}^{-1} \cdot \text{Hz}^{-1} \cdot \text{ster}^{-1}$$

Averaging over cosmic abundances, we find

$$j_\nu = 6.2 \times 10^{-39} \frac{n_e^2}{T^{1/2}} e^{-h\nu/kT} g_{ff} \ \text{erg} \cdot \text{cm}^{-3} \cdot \text{s}^{-1} \cdot \text{Hz}^{-1} \cdot \text{ster}^{-1}$$

And integrating over frequencies gives

$$P = 1.4 \times 10^{-27} Z^2 n_e n_i T^{1/2} \ \text{erg} \cdot \text{cm}^{-3} \cdot \text{s}^{-1}$$

Astrophysical sites where Bremmstrahlung is an important emission mechanism include HII regions ($T \sim$ few $\times 10^3$ K, so $kT/h \sim 1.4 \times 10^{-16}$ erg/K$\times$few$\times 10^3$ K$/ 6.6 \times 10^{-27}$ erg·s $= 6 \times 10^{13}$ Hz, so the emission is in the IR and the radio) and clusters of galaxies ($T \sim$ few$\times 10^7 - 10^8$ K, emission in the x-ray.)
. The cooling time is

$$
\begin{aligned}
\tau &= \frac{\frac{3}{2}\left(n_e + n_i\right)kT}{P} = \frac{3 \times \left(1.4 \times 10^{-16}\text{erg/K}\right) T\ (\text{K})}{1.4 \times 10^{-27} Z^2 n_e T^{1/2}\text{erg} \cdot \text{cm}^{-3} \cdot \text{s}^{-1}} \\
&= 3.0 \times 10^{11}\frac{T^{1/2}}{n}\ \text{s} = 10^4 \frac{T^{1/2}}{n}\ \text{s}
\end{aligned}
$$

If $T = 10^8$ K and $n = 10^{-2}$ cm^{-3}, as in clusters, then $\tau = 10^{10}$ yr, the age of the universe.

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