Green's function for the wave equation

Non-relativistic case

January 2020

1 The wave equations

In the **Lorentz Gauge**, the wave equations for the potentials are (Notes 1 eqns 43 and 44):

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{j} \tag{1}$$

and

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{\rho}{\varepsilon_0} \tag{2}$$

The Gauge condition is (Notes 1 eqn 42):

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{3}$$

In **Coulomb Gauge** we have the Gauge condition:

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{4}$$

which leads to the equations (Notes 1, eqn between 43 and 44, with $\vec{\nabla} \cdot \vec{A} = 0$)

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \tag{5}$$

and (Notes 1, eqn 41, with $\vec{\nabla} \cdot \vec{A} = 0$)

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{j} - \frac{1}{c^2}\vec{\nabla}\frac{\partial \Phi}{\partial t} \tag{6}$$

2 Longitudinal and transverse currents

The Coulomb Gauge wave equation for $\vec{A}$ (6) is awkward because it contains the scalar potential Φ. We can eliminate the potential Φ to obtain an equation for $\vec{A}$ alone. First we separate the current into two pieces, called the *longitudinal current* $\vec{J}_\ell$ and the *transverse current* $\vec{J}_t$:

$$\vec{J} = \vec{J}_\ell + \vec{J}_t$$

where

$$\vec{\nabla} \cdot \vec{J}_t = 0 \tag{7}$$

and

$$\vec{\nabla} \times \vec{J}_\ell = 0 \tag{8}$$

(Basically we are applying the Helmholtz theorem. We can always do this: See Lea Appendix II). From charge conservation (Notes 1 eqn 7):

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$$

and using equation (5) to eliminate ρ, this becomes:

$$\vec{\nabla}\cdot\vec{J}_\ell=-\frac{\partial}{\partial t}\left(-\varepsilon_0\nabla^2\Phi\right)=\vec{\nabla}\cdot\left[\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\Phi\right)\right]$$

From (8), $\vec{J}_\ell$ is the gradient of a scalar, so

$$\vec{J}_\ell=\varepsilon_0\vec{\nabla}\frac{\partial}{\partial t}\Phi \tag{9}$$

Using equation (9) in equation (6), we have:

$$\begin{aligned}\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\left(\vec{J}-\varepsilon_0\vec{\nabla}\frac{\partial\Phi}{\partial t}\right)\\ &= -\mu_0\left(\vec{J}-\vec{J}_\ell\right)=-\mu_0\vec{J}_t \qquad (10)\end{aligned}$$

$\boxed{\text{In the Coulomb Gauge, the transverse current } \vec{J}_t \text{ is the source of } \vec{A}.}$

We can also use result (9) to express $\vec{J}_l$ in terms of $\vec{J}$. Since equation (5) is the same as in the static case, the solution is also the same (Notes 1 eqn 29). Thus

$$\begin{aligned}\vec{J}_\ell\left(\vec{x},t\right) &= \frac{1}{4\pi}\vec{\nabla}\frac{\partial}{\partial t}\int\frac{\rho\left(\vec{x}',t\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'\\ &= \frac{1}{4\pi}\vec{\nabla}\int\frac{\frac{\partial}{\partial t}\rho\left(\vec{x}',t\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'\\ \vec{J}_\ell\left(\vec{x},t\right) &= -\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x' \qquad (11)\end{aligned}$$

where we used charge conservation in the last step. We can also express $\vec{J}_t$ in terms of $\vec{J}$ as follows, using (11),

$$\vec{J}_t=\vec{J}-\vec{J}_\ell=\vec{J}+\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}}{\left|\vec{x}-\vec{x}'\right|}d^3x'$$

Let's work on the integral. We start with an "integration by parts":

$$\begin{aligned}\int\frac{\vec{\nabla}'\cdot\vec{J}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x' &= \int\vec{\nabla}'\cdot\left(\frac{\vec{J}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}\right)d^3x'-\int\vec{J}\cdot\vec{\nabla}'\frac{1}{\left|\vec{x}-\vec{x}'\right|}\,d^3x'\\ &= \int_{S_\infty}\frac{\vec{J}\left(\vec{x}'\right)\cdot\hat{n}}{\left|\vec{x}-\vec{x}'\right|}d^2x'+\int\vec{J}\cdot\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{x}'\right|}\,d^3x' \quad \text{(Notes 1 eqn 19)}\\ &= 0+\vec{\nabla}\cdot\int\frac{\vec{J}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'\end{aligned}$$

The surface integral is zero provided that $\vec{J}$ is localized. Then

$$\begin{aligned}
\vec{J}_t(\vec{x},t) &= \vec{J}(\vec{x},t) + \frac{1}{4\pi}\vec{\nabla}\left(\vec{\nabla}\cdot\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d^3x'\right) \\
&= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla}\times\left(\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'\right) + \nabla^2\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'\right] \\
&= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla}\times\left(\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'\right) + \int\vec{J}\,\nabla^2\frac{1}{|\vec{x}-\vec{x}'|}d^3x'\right] \\
&= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla}\times\left(\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'\right) + \int\vec{J}\left[-4\pi\delta(\vec{x}-\vec{x}')\right]\,d^3x'\right] \\
\vec{J}_t(\vec{x},t) &= \frac{1}{4\pi}\vec{\nabla}\times\left(\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d^3x'\right) \qquad (12)
\end{aligned}$$

We still have the rather unphysical result from equation (5) that Φ changes instantaneously everywhere as ρ changes. In classical physics the potential is just a mathematical construct that we use to find the fields, and it can be shown (*e.g.* J prob 6.20) that $\vec{E}$ is causal even though Φ is not.

3 The Green's function

With either gauge we have a wave equation of the form

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \text{(source)}$$

where Φ may be either the scalar potential (in Lorentz Gauge) or a Cartesian component of $\vec{A}$. (In Coulomb Gauge the scalar potential is found using the methods we have already developed for the static case.) The corresponding Green's function problem is:

$$\nabla^2 G(\vec{x},t;\vec{x}',t') - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\vec{x}-\vec{x}')\,\delta(t-t') \qquad (13)$$

where the source is now a unit *event* located at position $\vec{x} = \vec{x}'$ and happening at time $t = t'$. As usual, the primed coordinates $\vec{x}'$, t', are considered fixed for the moment. To solve this equation we first Fourier transform in time (see, eg, Lea pg 503):

$$G(\vec{x},t;\vec{x}',t') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\vec{x},\omega;\vec{x}',t')\,e^{-i\omega t}d\omega$$

and the transformed equation (13) becomes

$$\begin{aligned}
\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G(\vec{x},\omega;\vec{x}',t') &= -4\pi\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(\vec{x}-\vec{x}')\,\delta(t-t')\,e^{i\omega t}dt \\
&= -\frac{4\pi}{\sqrt{2\pi}}\delta(\vec{x}-\vec{x}')\,e^{i\omega t'}
\end{aligned}$$

So let $G(\vec{x},\omega;\vec{x}',t') = g(\vec{x},\vec{x}')\, e^{i\omega t'}/\sqrt{2\pi}$ and then g satisfies the equation

$$\left(\nabla^2 + k^2\right) g = -4\pi\delta\left(\vec{x} - \vec{x}'\right) \tag{14}$$

where $k = \omega/c$. In free space without boundaries, g must be a function only of $R = |\vec{x} - \vec{x}'|$ and must posess spherical symmetry about the source point[1]. Thus in spherical coordinates with origin at the point P' with coordinates $\vec{x}'$, we can write:

$$\frac{1}{R}\frac{d^2}{dR^2}(Rg) + k^2 g = -4\pi\delta\left(\vec{R}\right) \tag{15}$$

For $\vec{R} \neq 0$, the right hand side is zero. Then the function Rg satisfies the exponential equation, and the solution is:

$$\begin{aligned} Rg &= Ae^{ikR} + Be^{-ikR} \\ g &= \frac{1}{R}\left(Ae^{ikR} + Be^{-ikR}\right) \qquad R \neq 0 \end{aligned} \tag{16}$$

Near the origin, where the delta-function contributes, the second term on the LHS of (14) is negligible compared with the first, and equation (14) becomes:

$$\nabla^2 g \simeq -4\pi\delta\left(\vec{x} - \vec{x}'\right)$$

We recognize that this equation has the solution (Lea eqn 6.26)

$$g = \frac{1}{R}$$

This is consistent with equation (16) as $R \to 0$, provided that

$$A + B = 1$$

Thus we have the solution

$$G(\vec{x},\omega;\vec{x}',t') = \frac{1}{\sqrt{2\pi}R}\left[Ae^{ikR} + (1-A)\,e^{-ikR}\right] e^{i\omega t'}$$

You should convince yourself that this solution is correct by differentiating and stuffing back into equation (15).

Now we do the inverse transform:

$$\begin{aligned} G(\vec{x},t;\vec{x}',t') &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}R}\left(Ae^{ikR} + Be^{-ikR}\right)e^{i\omega t'}e^{-i\omega t}d\omega \\ &= \frac{1}{2\pi R}\int_{-\infty}^{\infty}\left\{A\exp\left[i\omega\left(R/c + t' - t\right)\right] + B\exp\left[i\omega\left(-R/c + t' - t\right)\right]\right\}d\omega \\ &= \frac{A}{R}\,\delta\left[t' - (t - R/c)\right] + \frac{B}{R}\,\delta\left[t' - (t + R/c)\right] \end{aligned} \tag{17}$$

where we used Lea eqn 6.16. The second term is usually rejected (take $B \equiv 0$ and thus $A = 1$) because it predicts a response to an event occurring in the future. However, Feynman and Wheeler[2] have proposed a theory in which both terms are kept. They show that this theory can be consistent with observed causality provided that the universe is perfectly absorbing in the infinite future. (This now appears unlikely.) The time $t - R/c$

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[1] Note that the operator $\nabla^2 + k^2$ is spherically symmetric.
[2] Reviews of Modern Physics, 1949, **21**, 425

that appears in the first term is called the *retarded time* $t_{\text{ret.}}$ Thus we take

$$G(\vec{x},t;\vec{x}',t') = \frac{1}{R}\,\delta\left[t' - (t - R/c)\right] = \frac{1}{R}\,\delta\left(t' - t_{\text{ret}}\right) \tag{18}$$

Causality (an event cannot precede its cause) requires that the symmetry of this Green's function is:

$$G(\vec{x},t;\vec{x}',t') = G(\vec{x}',-t';\vec{x},-t)$$

(See Morse and Feshbach Ch 7 pg 834-835). and also

$$G(\vec{x},-\infty;\vec{x}',t') = 0$$

and

$$G(\vec{x},t;\vec{x}',t') = 0 \text{ for } t < t'$$

4 The potentials

Now that we have the Green's function (18), we can solve our original equations. Modifying eqn 1.44 in Jackson ("Formal" Notes eqn 7) to include time dependence, and with $S \to \infty$, we get an integral over a volume in space-time rather than just space:

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}',t')\,G(\vec{x},t;\vec{x}',t')\; dt'd^3\vec{x}'$$

Thus, inserting (18), we have

$$\begin{aligned}\Phi(\vec{x},t) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}',t')}{R}\delta\left(t' - t_{\text{ret}}\right)\; dt'd^3\vec{x}' &(19)\\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}',t_{\text{ret}})}{R(t_{\text{ret}})}\; d^3\vec{x}' &(20)\end{aligned}$$

in Lorentz Gauge, and similarly:

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{x}',t_{\text{ret}})}{R(t_{\text{ret}})}\; d^3x' \tag{21}$$

Notice that these equations have the same form as the static potentials (equations 1.17 and 5.32 in Jackson, eqns 21 and 29 in Notes 1), but we must evaluate the source and the distance R at the retarded time. This allows for the time for a signal to travel from the source to the observer at speed c. Note that t_{ret} is a function of both $\vec{x}$ and $\vec{x}'$, as well as t.

Similar equations hold in Coulomb Gauge. The scalar potential (Notes 1 eqn. 29) changes instantaneously as ρ changes, and the vector potential involves the transverse current only in equation (21). In spite of this peculiarity, the fields $\vec{E}$ and $\vec{B}$ are causal. (See problem 6.20 which demonstrates this in a relatively simple case. This problem may be "relatively" simple, but it is not easy.)

When spatial boundaries are present the analysis is more complicated. We must use the same kind of techniques that we used in the static case, expanding G in eigenfunctions. However, the vector nature of $\vec{A}$ makes the problem much harder. (See Chapter 9 sections 6-12.)

5 Radiation from a moving point charge

5.1 The Lienard-Wiechert potentials

Here the source is a point charge q with position $\vec{r}(t)$ that is moving with velocity $\vec{v}(t)$. The charge and current densities are

$$\rho(\vec{x},t) = q\delta[\vec{x} - \vec{r}(t)]$$

and

$$\vec{j}(\vec{x},t) = q\vec{v}\delta[\vec{x} - \vec{r}(t)]$$

Because the source terms are delta-functions, it turns out to be easier to back up one step. Then from equation (19), we have:

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{q\delta[\vec{x}' - \vec{r}(t')]}{R}\delta(t' - t_{\text{ret}})\ dt' d^3x'$$

We do the integral over the spatial coordinates first. Then

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int q\frac{\delta[t' + R(t')/c - t]}{R(t')}dt'$$

where $R(t') = |\vec{x} - \vec{r}(t')|$. To do the t' integral, we must re-express the delta-function. Recall (Lea eqn 6.10)

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i) \tag{22}$$

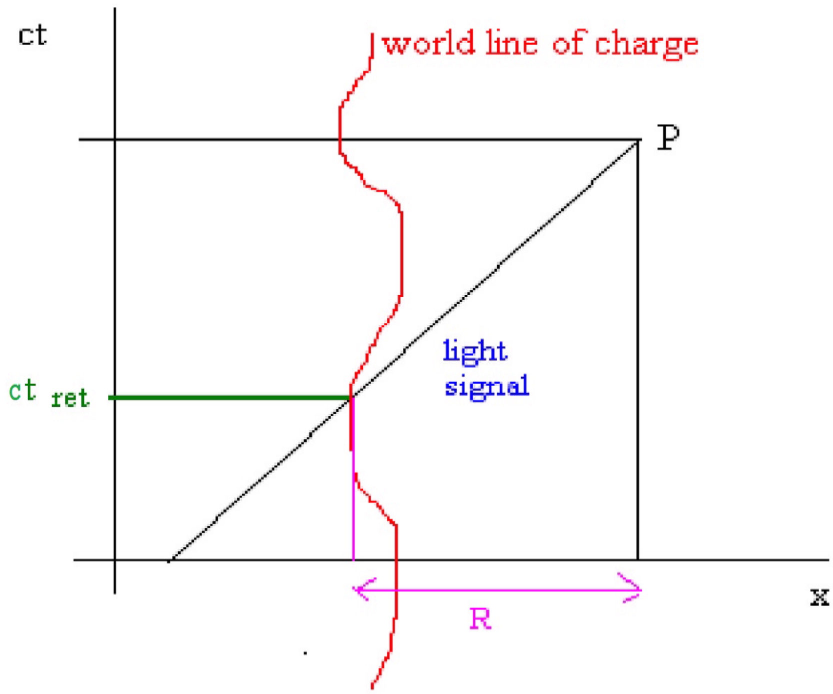
where $f(x_i) = 0$. In this case:

$$f(t') = t' + \frac{R(t')}{c} - t$$

and, since $\vec{v} = d\vec{r}/dt$,

$$\begin{aligned} f'(t') &= 1 + \frac{1}{c}\frac{dR}{dt'} = 1 + \frac{1}{c}\frac{d}{dt'}\sqrt{[\vec{x} - \vec{r}(t')]\cdot[\vec{x} - \vec{r}(t')]} \\ &= 1 + \frac{1}{c}\frac{[\vec{x} - \vec{r}(t')]}{|\vec{x} - \vec{r}(t')|}\cdot\frac{d}{dt'}[\vec{x} - \vec{r}(t')] \\ &= 1 - \frac{\vec{v}\cdot[\vec{x} - \vec{r}(t')]}{c|\vec{x} - \vec{r}(t')|} = 1 - \frac{\vec{v}\cdot\vec{R}}{cR} = 1 - \frac{\vec{v}\cdot\hat{R}}{c} \end{aligned} \tag{23}$$

The derivative f' is always positive, since $v < c$. The function f is zero when t' equals the solution of the equation $t_{\text{ret}} = t - R(t_{\text{ret}})/c$. A space-time diagram shows this most easily: t_{ret} is found from the intersection of the backward light cone from P with the charge's world line. There is only one root.



Thus, evaluating the integral using (22) and (23), we get:

$$\Phi\left(\vec{x},t\right) = \frac{1}{4\pi\varepsilon_0} \int q \frac{\delta\left(t' - t_{\text{ret}}\right)}{R\left(t'\right)\left(1 - \frac{\vec{v}\cdot\vec{R}}{cR}\right)} dt' = \frac{1}{4\pi\varepsilon_0} \left. \frac{q}{R\left(1 - \frac{\vec{v}\cdot\vec{R}}{cR}\right)} \right|_{t_{\text{ret}}} \tag{24}$$

and similarly

$$\vec{A}\left(\vec{x},t\right) = \frac{\mu_0}{4\pi} \left. \frac{q\vec{v}}{R\left(1 - \frac{\vec{v}\cdot\vec{R}}{cR}\right)} \right|_{t_{\text{ret}}} \tag{25}$$

These are the Lienhard-Wiechert potentials. It is convenient to use the shorthand

$$r_v = R\left(1 - \frac{\vec{v}\cdot\vec{R}}{cR}\right) = R - \frac{\vec{v}\cdot\vec{R}}{c} \tag{26}$$

so that

$$\vec{A}\left(\vec{x},t\right) = \frac{\mu_0}{4\pi} \left. \frac{q\vec{v}}{r_v} \right|_{t_{\text{ret}}} = \frac{\vec{v}}{c^2}\Phi \tag{27}$$

5.2 Calculating the fields

The fields are found using the usual relations (notes 1 eqns 40 and 20)

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

But our expressions for the potentials are in terms of $\vec{x}$ and t_{ret}, not $\vec{x}$ and t, so we have to be very careful in taking the partial derivatives. We can put the origin at the instantaneous position of the charge to simplify things. Then, using spherical coordinates, $R = r$. Our potential may be written:

$$\Phi\left(\vec{x}, t\right) \equiv \Psi\left(\vec{x}, t_{\text{ret}}\right)$$

A differential change in the potential due to a change in the coordinates is

$$d\Phi = \vec{\nabla}\Phi\Big|_{\text{const } t} \cdot d\vec{x} + \frac{\partial \Phi}{\partial t} dt \equiv \vec{\nabla}\Psi\Big|_{\text{const } t_{\text{ret}}} \cdot d\vec{x} + \frac{\partial \Psi}{\partial t_{\text{ret}}} dt_{\text{ret}} = d\Psi$$

But $dt_{\text{ret}} = dt - dr/c$, so

$$\vec{\nabla}\Phi\Big|_{\text{const } t} \cdot d\vec{x} + \frac{\partial \Phi}{\partial t} dt \equiv \vec{\nabla}\Psi\Big|_{\text{const } t_{\text{ret}}} \cdot d\vec{x} - \frac{\partial \Psi}{\partial t_{\text{ret}}} \frac{dr}{c} + \frac{\partial \Psi}{\partial t_{\text{ret}}} dt$$

This must be true for any $d\vec{x}$ and dt. Comparing the coefficient of dr on both sides, we see that the $r-$component of $\vec{\nabla}\Phi$ must be modified:

$$\frac{\partial \Phi}{\partial r}\Big|_{\text{const } t} = \frac{\partial \Psi}{\partial r}\Big|_{\text{const } t_{\text{ret}}} - \frac{1}{c} \frac{\partial \Psi}{\partial t_{\text{ret}}} \tag{28}$$

With this result, we can calculate the fields:

$$\vec{\nabla}\Phi = \frac{1}{4\pi\varepsilon_0} \vec{\nabla} \frac{q}{r_v} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{r_v^2} \vec{\nabla} r_v \tag{29}$$

where

$$\vec{\nabla} r_v = \frac{\partial}{\partial r}\left(r - \frac{\vec{r} \cdot \vec{v}}{c}\right)\hat{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}\left(r - \frac{\vec{r} \cdot \vec{v}}{c}\right) + \frac{\hat{\boldsymbol{\phi}}}{r \sin\theta}\frac{\partial}{\partial \phi}\left(r - \frac{\vec{r} \cdot \vec{v}}{c}\right)$$

We can choose our axes with polar axis along the instantaneous direction of $\vec{v}$. Then $\vec{r} \cdot \vec{v} = rv\cos\theta$, and

$$\vec{\nabla} r_v = \left(1 - \frac{v}{c}\cos\theta\right)\hat{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\left(r\frac{v}{c}\sin\theta\right)$$

In this coordinate system

$$\vec{v} = v\hat{z} = v\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$$

so

$$\vec{\nabla} r_v = \hat{r} - \frac{\vec{v}}{c} \tag{30}$$

In the non-relativistic limit, $v/c \ll 1$, to zeroth order in v/c, this becomes $\vec{\nabla} r_v = \hat{r}$. We are also going to need

$$\frac{\partial r_v}{\partial t} = -\frac{\vec{r} \cdot \vec{a}}{c} \tag{31}$$

Then, using (27), we have

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\Phi\Big|_{\text{const t}_{\text{ret}}} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\hat{r} - \frac{\partial \vec{A}}{\partial t} \\ &= -\vec{\nabla}\Phi\Big|_{\text{const t}_{\text{ret}}} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\hat{r} - \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{v}\Phi) \\ &= -\vec{\nabla}\Phi\Big|_{\text{const t}_{\text{ret}}} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\left(\hat{r} - \frac{\vec{v}}{c}\right) - \frac{\Phi}{c^2}\frac{\partial}{\partial t}\vec{v}\end{aligned}$$

Inserting our results (29) , (30) and (31), we get

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{r_v^2}\left(\hat{r} - \frac{\vec{v}}{c}\right) - \frac{(\hat{r} - \vec{v}/c)}{c}\frac{q}{r_v^2}\left(-\frac{\vec{r}\cdot\vec{a}}{c}\right) - \frac{q}{r_v}\frac{\vec{a}}{c^2}\right\}$$

Combining the terms that involve the acceleration, we have

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_v^2}\left(\hat{r} - \frac{\vec{v}}{c}\right) + \frac{q}{4\pi\varepsilon_0 r_v c^2}\frac{(\hat{r} - \vec{v}/c)(\hat{r}\cdot\vec{a}) - \vec{a}(1 - \hat{r}\cdot\vec{v}/c)}{(1 - \hat{r}\cdot\vec{v}/c)} \tag{32}$$

$$= \frac{q}{4\pi\varepsilon_0 r_v^2}\left(\hat{r} - \frac{\vec{v}}{c}\right) + \frac{\mu_0 q}{4\pi r_v(1 - \hat{r}\cdot\vec{v}/c)}\hat{r}\times\left[\left(\hat{r} - \frac{\vec{v}}{c}\right)\times\vec{a}\right] \tag{33}$$

Taking the non-relativistic limit[3] $v/c \ll 1$, (33) becomes:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r^2}\hat{r} + \frac{q}{c^2}\frac{[\hat{r}\times(\hat{r}\times\vec{a})]}{r}\right]$$

The first term is the usual Coulomb field which goes as $1/r^2$. The second term depends on the acceleration $\vec{a}$: this is the radiation field.

$$\vec{E}_{\text{rad}} = \frac{\mu_0}{4\pi}\frac{q}{r}[\hat{r}\times(\hat{r}\times\vec{a})] \tag{34}$$

This term decreases as $1/r$ and dominates at large r. Note also that $\vec{E}_{\text{rad}}$ is perpendicular to $\hat{r}$.

Next let's calculate the magnetic field:

$$\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A}\Big|_{\text{const } t} \\ &= \left(\vec{\nabla}\Big|_{\text{const } t_{\text{ret}}} - \frac{1}{c}\hat{r}\frac{\partial}{\partial t}\right)\times\frac{\mu_0}{4\pi}\frac{q\vec{v}}{r_v} \\ \vec{B} &= \frac{\mu_0}{4\pi}q\left[\vec{\nabla}\left(\frac{1}{r_v}\right)\times\vec{v} - \frac{1}{c}\hat{r}\times\left(\frac{\vec{a}}{r_v} - \frac{\vec{v}}{r_v^2}\frac{\partial r_v}{\partial t}\right)\right] \\ &= -\frac{\mu_0}{4\pi}q\left[\frac{1}{r_v^2}\left(\hat{r} - \frac{\vec{v}}{c}\right)\times\vec{v} - \frac{\hat{r}}{c}\times\left(\frac{\vec{a}}{r_v} + \frac{\vec{v}}{r_v^2}\frac{\vec{r}\cdot\vec{a}}{c}\right)\right] \\ &= \frac{\mu_0 q}{4\pi r_v^2}\vec{v}\times\hat{r} - \frac{\mu_0}{4\pi}\frac{q/c}{r_v(1 - \vec{v}\cdot\hat{r}/c)}\left[(\hat{r}\times\vec{a})(1 - \vec{v}\cdot\hat{r}/c) + (\hat{r}\times\vec{v})\frac{\hat{r}\cdot\vec{a}}{c}\right]\end{aligned}$$

[3] Eqn (33) is not quite correct if v/c is not small, as we have not done a correct relativistic treatment of time. We are missing some factors of γ and $1 - \vec{\beta}\cdot\hat{r}$.

We can simpify this result using (32) for $\vec{E}_{\text{rad}}$ and the fact that $\hat{r} \times \hat{r} \equiv 0$.

$$\begin{aligned} \vec{B} &= \frac{\mu_0 q}{4\pi r_v^2} \vec{v} \times \hat{r} + \frac{\mu_0}{4\pi} \frac{q}{r_v (1 - \vec{v} \cdot \hat{r}/c)} \frac{\hat{r} \times [(\hat{r} - \vec{v}/c)(\hat{r} \cdot \vec{a}) - \vec{a}(1 - \hat{r} \cdot \vec{v}/c)]}{c} \\ &= \vec{B}_{\text{B-S}} + \hat{r} \times \vec{E}_{\text{rad}}/c \end{aligned} \tag{35}$$

In the limit $v/c \ll 1$, the first term, which goes as $1/r^2$, is the usual Biot-Savart law result . The second term, which goes as $1/r$, is the radiation field. Notice that

$$\vec{B}_{\text{rad}} = \hat{r} \times \vec{E}_{\text{rad}}/c$$

as expected for an EM wave in free space. In the non-relativistic limit,

$$\vec{B}_{\text{rad}} = \frac{\mu_0}{4\pi} \frac{q}{rc} \vec{a} \times \hat{r} \tag{36}$$

5.3 Radiated Power

The Poynting flux for the radiation field is:

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0} \vec{E}_{\text{rad}} \times \vec{B}_{\text{rad}} = \frac{1}{\mu_0} \vec{E}_{\text{rad}} \times \left(\frac{\hat{r} \times \vec{E}_{\text{rad}}}{c} \right) \\ &= \frac{E_{\text{rad}}^2}{\mu_0 c} \hat{r} \end{aligned} \tag{37}$$

where from equation (34):

$$E_{\text{rad}} = \frac{\mu_0}{4\pi} \frac{q}{r} a \left| \sin\theta \right| = \frac{1}{4\pi\varepsilon_0} \frac{q}{rc^2} a \left| \sin\theta \right|$$

and θ is the angle between $\vec{a}$ and $\hat{r}$. Thus, in the non-relativistic limit

$$S = \frac{1}{\mu_0 c} \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{rc^2} a \sin\theta \right)^2 = \frac{q^2}{(4\pi)^2 \varepsilon_0 c^3} \frac{a^2}{r^2} \sin^2\theta$$

Notice that $S \propto 1/r^2$, the usual inverse square law for light. S is the power radiated per unit area of wavefront, $S = dP/dA$. Writing dA in terms of solid angle, $dA = r^2 d\Omega$, we find the power radiated per unit solid angle is independent of distance:

$$\frac{dP}{d\Omega} = r^2 S = \frac{q^2 a^2}{(4\pi)^2 \varepsilon_0 c^3} \sin^2\theta \tag{38}$$

Finally the total power radiated is

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{q^2 a^2}{(4\pi)^2 \varepsilon_0 c^3} \int_0^{2\pi} \int_{-1}^{+1} \left(1 - \mu^2\right) d\phi d\mu$$

where as usual $\mu = \cos\theta$. Thus

$$\begin{aligned} P &= \frac{q^2 a^2}{(4\pi\varepsilon_0)\, 2c^3} \left(\mu - \frac{\mu^3}{3} \right) \Big|_{-1}^{+1} \\ P &= \frac{2}{3} \frac{q^2 a^2}{4\pi\varepsilon_0 c^3} \end{aligned} \tag{39}$$

This result is called the Larmor formula.

In the relativistic case, we use (33) in (37) to get:

$$\vec{S} = \frac{1}{\mu_0 c}\,\hat{r}\left|\frac{\mu_0 q}{4\pi r_v\left(1-\hat{r}\cdot\vec{v}/c\right)}\hat{r}\times\left[\left(\hat{r}-\frac{\vec{v}}{c}\right)\times\vec{a}\right]\right|^2$$

and

$$\frac{dP}{d\Omega} = \frac{1}{c}\frac{\mu_0 q^2}{\left(4\pi\right)^2\left(1-\hat{r}\cdot\vec{v}/c\right)^4}\left|\hat{r}\times\left[\left(\hat{r}-\frac{\vec{v}}{c}\right)\times\vec{a}\right]\right|^2$$

The denominator $(1-\hat{r}\cdot\vec{v}/c)^4$ indicates that the radiation is beamed into a small cone around the velocity vector when $v/c \to 1$. Note here that this derivation is not strictly correct when v/c is not negligible, and it leads to the wrong power of $(1-\hat{r}\cdot\vec{v}/c)$ in the denominator. It does indicate qualitatively how the radiation is beamed. For the correct relativistic derivation see Jackson Ch 14 or http://www.physics.sfsu.edu/~lea/courses/grad/radgen.PDF.

Example

A particle of charge q and mass m is moving in the presence of a uniform magnetic field $\vec{B} = B_0\hat{z}$. Its speed $v \ll c$. Find the power radiated.

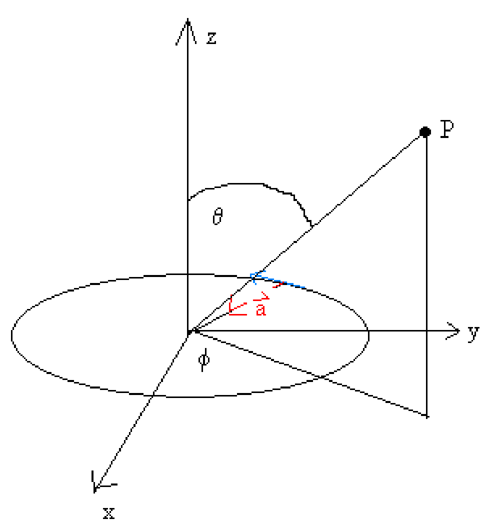
First we compute the acceleration:

$$\vec{F} = m\vec{a} = q\vec{v}\times\vec{B}$$

so

$$\vec{a} = \frac{q}{m}\vec{v}\times\vec{B} = -\frac{q}{m}\vec{B}\times\vec{v} = \vec{\omega}\times\vec{v} \tag{40}$$

Only the component of $\vec{v}$ perpendicular to $\vec{B}$ contributes, and the motion is a circle with $\vec{a}$ pointing toward the center. If the component of $\vec{v}$ parallel to $\vec{B}$ is not zero, the motion is a helix. $v_{\|}$ remains unchanged and does not affect the radiation if $v \ll c$, so it will be ignored from now on. (There are important beaming effects if v/c is NOT $\ll 1$.)



Choose coordinates as shown in the diagram above, and choose $t = 0$ when the particle is on the $x-$axis. Then the angle χ between $\hat{r}$ and $\vec{a}$ is found from

$$\hat{r} \cdot \hat{a} = \cos\chi = \sin\theta\cos(\omega t - \phi)$$

where

$$\omega = \left|\frac{qB}{m}\right|$$

is the cyclotron frequency (eqn 40). Thus from (38),

$$\frac{dP}{d\Omega} = \frac{q^2}{(4\pi)^2\,\varepsilon_0 c^3}\left(\frac{q}{m}v_\perp B_0\right)^2\left(1 - \sin^2\theta\cos^2(\omega t - \phi)\right)$$

Taking the time average, we get

$$< \frac{dP}{d\Omega} > \;= \frac{q^4\,(v_\perp B_0)^2}{m^2\,(4\pi)^2\,\varepsilon_0 c^3}\left(1 - \frac{\sin^2\theta}{2}\right) = \frac{q^4 B_0^2 K_\perp}{16\pi^2\varepsilon_0 m^3 c^3}\left(1 + \cos^2\theta\right)$$

where $K_\perp$ is the particle's kinetic energy $\frac{1}{2}mv_\perp^2$. Radiation is maximum along the direction of $\vec{B}$, and minimum in the plane perpendicular to $\vec{B}$. The total power radiated is (39):

$$P = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0 c^3}\left(\frac{q}{m}v_\perp B_0\right)^2 = \frac{2}{3}\frac{q^4\,(v_\perp B_0)^2}{4\pi\varepsilon_0 c^3 m^2} = \frac{q^4 B_0^2 K_\perp}{3\pi\varepsilon_0 c^3 m^3}$$

The radiated power is proportional to the square of the magnetic field strength.

We can check the physical dimensions:

$$P = \frac{q^4\mu_0 B_0^2}{6\pi\varepsilon_0\mu_0 m^2 c}\left(\frac{v_\perp}{c}\right)^2 = \frac{q^4 c u_B}{3\pi\varepsilon_0^2\,(mc^2)^2}\left(\frac{v_\perp}{c}\right)^2$$

Since the energy of a pair of point charges is

$$U = \frac{q^2}{4\pi\varepsilon_0 d}$$

we have

$$P = (\text{energy} \times \text{length})^2\,\frac{\text{length}}{\text{time}}\,\frac{\text{energy}}{\text{volume}}\,\frac{1}{(\text{energy})^2} = \frac{\text{energy}}{\text{time}}$$

as required.

Things get much more interesting as $v \to c$. Synchrotron radiation is discussed in J Ch 14 and also in http://www.physics.sfsu.edu/~lea/courses/grad/radiation.PDF.