Spherical harmonics 2020

1 Problems with spherical symmetry: spherical harmonics

Suppose our potential problem has spherical boundaries. Then we would like to solve the problem in spherical coordinates. Let's look at Laplace's equation again.

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

We apply the same techniques that we used in the rectangular problem; only the details change. Look for a solution of the form

$$\Phi = R(r)\,P(\theta)\,W(\phi)$$

Then substituting in, and dividing by Φ, we get:

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P} + \frac{1}{Wr^2\sin^2\theta}\frac{\partial^2 W}{\partial\phi^2} = 0$$

To separate out an equation for $W(\phi)$, multiply the whole equation by $r^2\sin^2\theta$:

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P} + \frac{1}{W}\frac{\partial^2 W}{\partial\phi^2} = 0$$

Now the last term is a function of ϕ only, while the sum of the first two is a function of r and θ only. Thus if the solution is to satisfy the differential equation for *all* values of r, θ and ϕ, each of these two pieces must equal a constant.

Now if our region of interest is the inside or outside of a complete sphere, an increase of ϕ by any integer multiple of 2π corresponds to the same physical point. Thus the function Φ must have the same value for $\phi = \phi_1$ and $\phi = \phi_1 + 2\pi$, that is, the function W must be periodic with period 2π. We may achieve this behavior if we choose the separation constant so that

$$\frac{1}{W}\frac{\partial^2 W}{\partial\phi^2} = -m^2$$

with m equal to an integer. Then the solutions are the periodic functions:

$$W = \left\{\begin{array}{l}\sin m\phi \\ \cos m\phi\end{array}\right. \quad \text{or } e^{\pm im\phi}$$

The equation in r and θ then becomes:

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P} - m^2 = 0$$

Next, to separate the r and θ dependences, we divide through by $\sin^2\theta$, to get:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P}-\frac{m^2}{\sin^2\theta}=0$$

The first term is a function of r only while the sum of the last two is a function of θ only. Thus again both pieces must be constant. The equation has separated. Let

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=k \tag{1}$$

Then

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P}-\frac{m^2}{\sin^2\theta}+k=0$$

When working in spherical coordinates, changing variables to $\mu=\cos\theta$ is often a useful trick. Then $d\mu=-\sin\theta d\theta$, and the $\theta-$equation becomes:

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dP}{d\mu}\right]-\frac{m^2}{1-\mu^2}P+kP=0 \tag{2}$$

Equation (2) is known as the associated Legendre equation. Let's first tackle a special case.

1.1 Problems with axisymmetry: the Legendre polynomials

If the problem has rotational symmetry about the polar axis, then the function W must be a constant (Φ is independent of ϕ) and so $m=0$. Then equation 2 simplifies:

$$\frac{d}{d\mu}\left(\left(1-\mu^2\right)\frac{dP}{d\mu}\right)+kP=0 \tag{3}$$

We can solve this Legendre equation by looking for a series solution[1]. The singular points of the equation are at $\mu=\pm1$, so we should be able to find a solution about $\mu=0$ of the form:

$$y=\sum_{n=0}^{\infty}a_n\mu^n$$

Substituting into the equation, we have:

$$\sum_{n=0}^{\infty}n\left(n-1\right)a_n\mu^{n-2}-\sum_{n=0}^{\infty}n\left(n-1\right)a_n\mu^n-2\sum_{n=0}^{\infty}na_n\mu^n+k\sum_{n=0}^{\infty}a_n\mu^n=0$$

where each power of μ must separately equal zero. The constant term in the equation is:

$$2a_2+ka_0=0\Rightarrow a_2=-\frac{k}{2}a_0$$

[1] cf Lea Chapter 3 section 3.3.

and the first power of μ has coefficient:

$$3 \times 2a_3 - 2a_1 + ka_1 = 0 \Rightarrow a_3 = a_1 \frac{2-k}{3 \times 2}$$

For all higher powers, every term in the equation contributes. Looking at μ^p, setting $n = p + 2$ in the first term and $n = p$ in the rest, we find

$$(p+2)(p+1) a_{p+2} - p(p-1) a_p - 2pa_p + ka_p = 0$$

and so the recursion relation is:

$$a_{p+2} = a_p \frac{p(p-1) + 2p - k}{(p+2)(p+1)} = a_p \frac{p(p+1) - k}{(p+2)(p+1)} \tag{4}$$

The first two relations we obtained can also be described by this formula with $p = 0$ and $p = 1$ respectively. Since the recursion relation relates a_{p+2} to a_p, the soutions are purely even (starting with a_0) or purely odd (starting with a_1).

The solution we have obtained is valid for $-1 < \mu < 1$, but the series does not converge for $\mu = \pm 1$. This is a problem since $\mu = +1$ corresponds to $\theta = 0$ and $\mu = -1$ to $\theta = \pi$. These points are on the polar axis where usually we do not expect the potential to blow up. Thus we need a solution that remains valid up to and including these points. We can solve the problem by choosing the separation constant k so that the series terminates after a finite number of terms. In particular, if we choose k to have the value

$$k = l(l+1)$$

for some integer l, then according to the recursion relation (4):

$$a_{l+2} = a_l \frac{l(l+1) - l(l+1)}{(l+2)(l+1)} = 0$$

and so every succeeding a_p for $p > l$ is also zero. The corresponding solution is the Legendre Polynomial $P_l(\mu)$. By convention, we choose a_0 (for even l) or a_1(for odd l) so that

$$P_l(1) \equiv 1 \tag{5}$$

The recursion relation becomes:

$$a_{p+2} = a_p \frac{p(p+1) - l(l+1)}{(p+2)(p+1)} \tag{6}$$

The first few polynomials are:

$l = 0$: The only non-zero coefficient is a_0, which must equal 1 to make $P_0(1) = 1$, so:

$$P_0(\mu) = 1 \tag{7}$$

$l = 1$: The only non-zero coefficient is a_1, and again we must take $a_1 = 1$ to make $P_1(1) = 1$. Thus:

$$P_1(\mu) = \mu \tag{8}$$

$l = 2$:

$$a_2 = a_0 \left(\frac{-2 \times 3}{2} \right) = -3a_0$$

and subsequent a_n are all zero. Then:

$$P_2(\mu) = a_0 \left(1 - 3\mu^2\right)$$

and evaluating this at $\mu = 1$,we find

$$P_2(1) = a_0(-2) = 1 \Rightarrow a_0 = -\frac{1}{2}$$

Thus

$$P_2(\mu) = \frac{1}{2}\left(3\mu^2 - 1\right) \tag{9}$$

Notice the pattern: we use the recursion relation to determine the non-zero coefficients as multiples of the leading coefficient (a_0 or a_1). Then we evaluate the resulting polynomial at $\mu = 1$ and set the result equal to 1, thus determining the value of the leading coefficient.

Let's do one more:

$l = 3$: Applying the recursion relation (6) with $l = 3$ we find:

$$P_3(\mu) = a_1 \left(\mu + \frac{1.2 - 3.4}{3.2} \mu^3 \right)$$

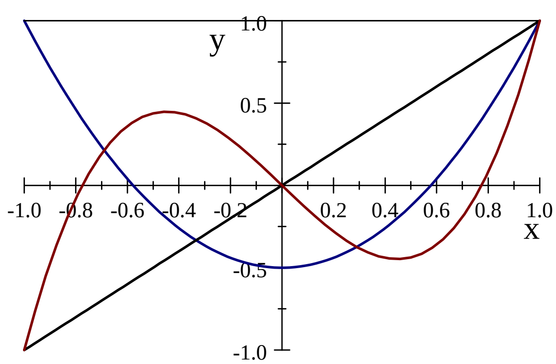
and evaluating at $\mu = 1$ gives:

$$P_3(1) = a_1 \left(1 - \frac{5}{3} \right) = 1 \Rightarrow a_1 = -\frac{3}{2}$$

and so

$$P_3(\mu) = \frac{\mu}{2}\left(5\mu^2 - 3\right) \tag{10}$$

The first four polynomials are shown in the figure.



P_0, P_1, P_2, and P_3

1.2 Solution for the potential

Now that we have the function of θ, let's return to the potential problem and solve for the function of r. With the separation constant determined, equation (1) becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=l\left(l+1\right)R$$

Solutions to this equation are powers of $r : R = r^p$ where

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial r^p}{\partial r}\right)=\frac{\partial}{\partial r}\left(r^2pr^{p-1}\right)=p\left(p+1\right)r^p=l\left(l+1\right)r^p$$

Thus one solution has $p = l$. There is a second solution with $p = -\left(l+1\right)$. Then $p+1 = -l$, and $p\left(p+1\right) = l\left(l+1\right)$ as required. Thus we have

$$R = r^l \text{ or } \frac{1}{r^{l+1}} \tag{11}$$

Thus an axisymmetric potential may be expressed as

$$\Phi\left(r,\theta\right)=\sum_{l=0}^{\infty}\left(A_lr^l+\frac{B_l}{r^{l+1}}\right)P_l\left(\mu\right) \tag{12}$$

where the constants A_l and B_l must be determined by the boundary conditions in r.

1.3 Orthogonality of the Legendre functions.

The Legendre equation (3) is of the Sturm-Liouville form (slreview notes eqn 1) with

$$f\left(\mu\right)\equiv 1-\mu^2$$

$$g\left(\mu\right)\equiv 0$$

and

$$w\left(\mu\right)\equiv 1$$

The eigenvalue is $\lambda = k = l\left(l+1\right)$. Even without specifying any boundary conditions, the Legendre functions must be orthogonal on the range $[-1,1]$ because $f\left(1\right) = f\left(-1\right) = 0$.

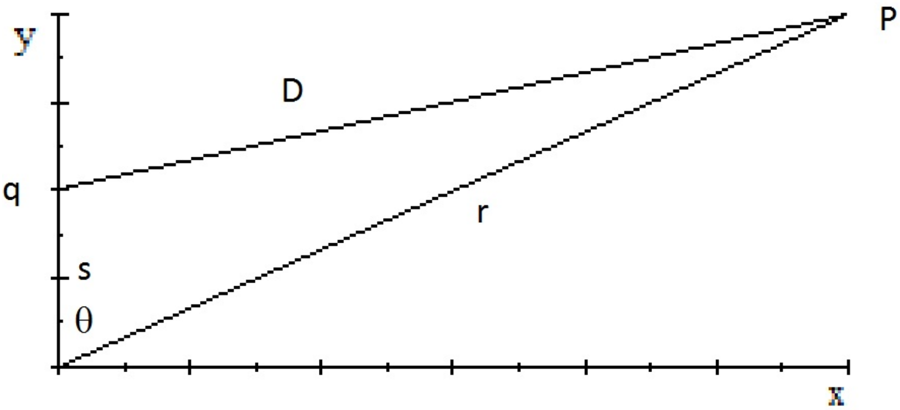
$$\int_{-1}^{+1}P_l\left(\mu\right)P_{l'}\left(\mu\right)d\mu=0 \text{ for } l\neq l' \tag{13}$$

To make use of this relation in forming series expansions in Legendre polynomials, we will need to find the value of the integral for $l = l'$. In the next few sections we shall collect some useful tools that will allow us to do that integral.

1.4 Properties of Legendre polynomials

1.4.1 The generating function

Suppose we put a point charge q on the polar axis at a distance s from the origin (See figure). Then the potential[2] at point P is



$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{D} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{s^2+r^2-2rs\cos\theta}}$$

which we can also express in the form (12). Now we let $x = s/r$ for convenience, and then for $r > s$, we can expand the function to get:

$$\begin{aligned}
\Phi &= \frac{q}{4\pi\varepsilon_0 r}\frac{1}{\sqrt{1+\frac{s^2}{r^2}-2\frac{s}{r}\mu}} = \frac{q}{4\pi\varepsilon_0}\left(1+x^2-2x\mu\right)^{-1/2} \\
&= \frac{q}{4\pi\varepsilon_0 r}\left(1-\frac{x^2-2x\mu}{2}+\frac{(-1/2)(-3/2)}{2}\left(x^2-2x\mu\right)^2+\cdots\right) \\
&= \frac{q}{4\pi\varepsilon_0 r}\left(1+x\mu-\frac{x^2}{2}\left(1-3\mu^2\right)+\cdots\right) \\
&= \frac{q}{4\pi\varepsilon_0 r}\left(1+xP_1(\mu)+x^2P_2(\mu)+\cdots\right)
\end{aligned}$$

which has the form (12) with $B_l = \frac{qs^l}{4\pi\varepsilon_0}$ for each l and $A_l \equiv 0$. Thus we have the identity:

$$\frac{1}{\sqrt{1-2x\mu+x^2}} = \sum_{l=0}^{\infty} x^l P_l(\mu) \tag{14}$$

[2] See, e.g., Lea and Burke Chapter 25, equation 25.9.

We can extend this result to find the potential for a point charge off axis, by letting γ be the angle between $\vec{x}$ and $\vec{x}'$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^l}P_l\left(\cos\gamma\right) \tag{15}$$

where $r_< = \min\left(r, r'\right)$ and $r_> = \max\left(r, r'\right)$.

The function

$$G\left(x,\mu\right) \equiv \frac{1}{\sqrt{1-2x\mu+x^2}} \tag{16}$$

is called the *generating function* for the Legendre polynomials. We can use it to determine several useful properties of the polynomials.

1.4.2 The orthogonality integral

We can obtain the integral (17) with $l = l'$ by integrating the square of the generating function:

$$\begin{aligned}\int_{-1}^{+1} G^2 d\mu &= \int_{-1}^{+1}\frac{1}{1-2x\mu+x^2}d\mu = \int_{-1}^{+1}\sum_{l=0}^{\infty}x^l P_l\left(\mu\right)\sum_{l'=0}^{\infty}x^{l'}P_{l'}\left(\mu\right)d\mu \\ &= \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}x^{l+l'}\int_{-1}^{+1}P_l\left(\mu\right)P_{l'}\left(\mu\right)d\mu\end{aligned}$$

The integral of the P_ls is zero unless $l = l'$. Thus, evaluating the integral of G^2 by a change of variable to $v = 1-2x\mu+x^2$, we have:

$$\begin{aligned}\frac{1}{-2x}\int_{(1+x)^2}^{(1-x)^2}\frac{dv}{v} &= \sum_{l=0}^{\infty}x^{2l}\int_{-1}^{+1}P_l\left(\mu\right)P_l\left(\mu\right)d\mu \\ &= \frac{1}{2x}\ln\frac{\left(1+x\right)^2}{\left(1-x\right)^2} = \frac{1}{x}\ln\frac{1+x}{1-x}\end{aligned}$$

Now since $x < 1$, we may expand the logarithm:

$$\begin{aligned}\frac{1}{x}\ln\frac{1+x}{1-x} &= \frac{2}{x}\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2l+1}}{2l+1}+\cdots\right) \\ &= 2\left(1+\frac{x^2}{3}+\cdots\frac{x^{2l}}{2l+1}+\cdots\right) = \sum_{l=0}^{\infty}x^{2l}\int_{-1}^{+1}P_l\left(\mu\right)P_l\left(\mu\right)d\mu\end{aligned}$$

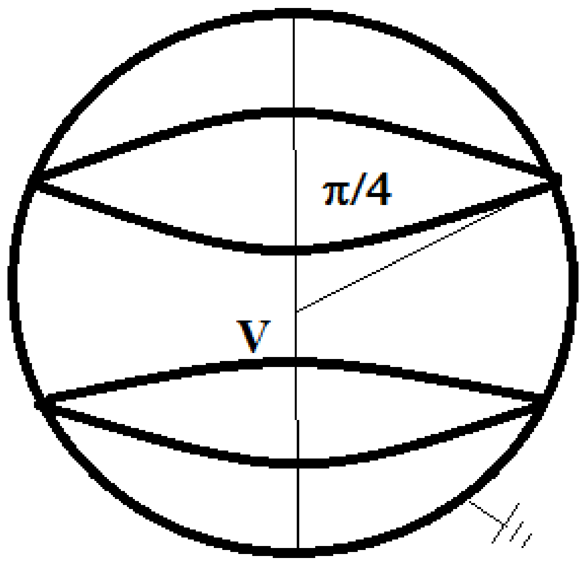
Both sides of this equation contain only even powers of x, and equating the coefficients of each power,we have:

$$\int_{-1}^{+1}P_l\left(\mu\right)P_l\left(\mu\right)d\mu = \frac{2}{2l+1} \tag{17}$$

which is the desired result.

1.5 Problem:

A conducting sphere is divided into three pieces by thin insulating strips at $\theta = \pi/4$, $3\pi/4$, as shown in the diagram. The polar regions are grounded and the equatorial region has potential V. Find the potential outside the sphere.



Model:

What do you think the field will look like at a great distance from the sphere? Why? (A point charge, because the area at potential V is greater than the grounded area, so I expect a net charge on the sphere.)

The system has rotational symmetry about the polar axis, drawn as shown in the diagram. It also has reflection symmetry about the equator.

Set-up:

Outside the sphere, the potential satisfies Laplace's equation, (all the charge is on the surface) and we expect $\Phi \to 0$ as $r \to \infty$, so there are no positive powers of r :

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\mu)$$

On the surface at $r = a$

$$\sum_{l=0}^{\infty} A_l a^l P_l(\mu) = \begin{matrix} 0 & if & 1 \geq \mu > \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}} > \mu \geq -1 \\ V & if & \frac{1}{\sqrt{2}} > \mu > -\frac{1}{\sqrt{2}} \end{matrix}$$

We will use othogonality of the $P_l(\mu)$ to find the cofficients A_l.

Solve:

We use Lea 8.39 (valid for $l > 0$):

$$\begin{aligned}
\frac{A_l}{a^{l+1}}\frac{2}{2l+1} &= V\int_{-1/\sqrt{2}}^{+1/\sqrt{2}} P_l(\mu)\,d\mu \qquad (18)\\
&= V\int_{-1/\sqrt{2}}^{+1/\sqrt{2}} \frac{P'_{l+1}(\mu) - P'_{l-1}(\mu)}{2l+1} d\mu \\
2\frac{A_l}{a^{l+1}} &= V\left(P_{l+1}(\mu) - P_{l-1}(\mu)\right)\Big|_{-1/\sqrt{2}}^{+1/\sqrt{2}}
\end{aligned}$$

Now

$$P_l(-\mu) = (-1)^l P_l(\mu),$$

so only terms with $l+1$ odd (l even) give non-zero results, so

$$A_l = Va^{l+1}\left[P_{l+1}\left(\frac{1}{\sqrt{2}}\right) - P_{l-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

We can simplify this using Lea 8.40 (valid for $l > 0$) and 8.41:

$$\begin{aligned}
A_l &= -Va^{l+1}\left(1-\frac{1}{2}\right)P'_l\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{l+1}+\frac{1}{l}\right)\\
&= -\frac{V}{2}a^{l+1}P'_l\left(\frac{1}{\sqrt{2}}\right)\frac{2l+1}{l(l+1)}
\end{aligned}$$

We must treat $l = 0$ separately. Returning to eqn (18) with $P_0(\mu) = 1$, we get

$$\begin{aligned}
2\frac{A_0}{a} &= V\int_{-1/\sqrt{2}}^{+1/\sqrt{2}} d\mu = \frac{2}{\sqrt{2}}V\\
A_0 &= \frac{Va}{\sqrt{2}}
\end{aligned}$$

Thus

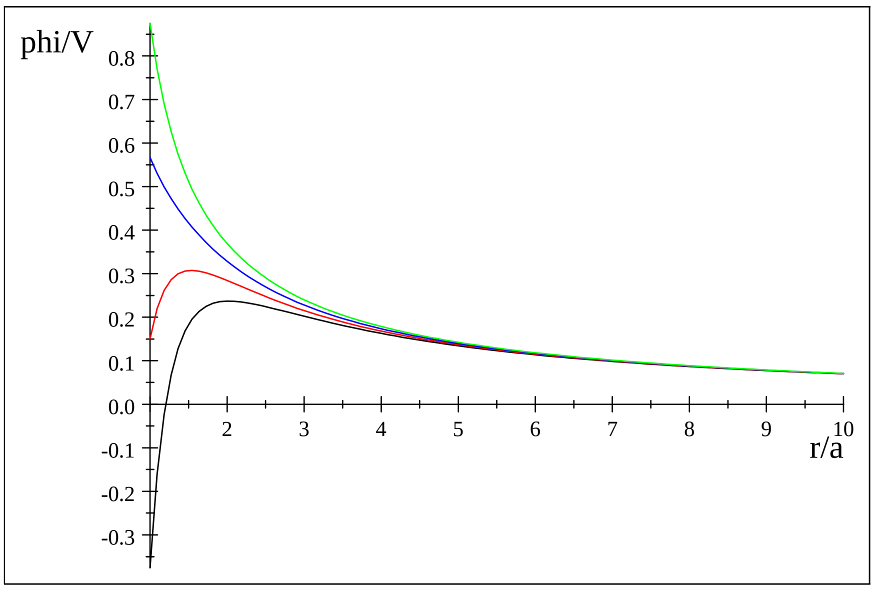
$$\begin{aligned}
\Phi(r,\theta) &= \frac{V}{\sqrt{2}}\frac{a}{r} - \frac{V}{2}\sum_{l=1}^{\infty}\left(\frac{a}{r}\right)^{2l+1}\frac{4l+1}{2l(2l+1)}P'_{2l}\left(\frac{1}{\sqrt{2}}\right)P_{2l}(\mu)\\
&= \frac{V}{\sqrt{2}}\frac{a}{r} - \frac{V}{4}\sum_{l=1}^{\infty}\left(\frac{a}{r}\right)^{2l+1}\frac{4l+1}{l(2l+1)}P'_{2l}\left(\frac{1}{\sqrt{2}}\right)P_{2l}(\mu)
\end{aligned}$$

Analysis:

The result is dimensionally correct. As expected, at large distances $r/a \gg 1$, we have the potential due to a point charge of magnitude $Q = 4\pi\varepsilon_0 Va/\sqrt{2}$. This is the net charge put onto the sphere by the battery system. The next term is a quadrupole, also as expected. We have only even l, which indicates the reflection symmetry about the equator. The series converges quite well due to the coefficient A_l which is of order $1/l$ for large l.

The first few terms are:

$$\begin{aligned}
\Phi(r,\theta) &= \frac{V}{\sqrt{2}}\frac{a}{r} - \frac{V}{4}\left\{\left(\frac{a}{r}\right)^3 \frac{5}{3}\left(\frac{3}{\sqrt{2}}\right)\left(\frac{3\mu^2-1}{2}\right) + \left(\frac{a}{r}\right)^5 \frac{9}{2(5)}\left(\frac{35}{2}\frac{1}{\sqrt{2}^3} - \frac{15}{2}\frac{1}{\sqrt{2}}\right)\frac{(35\mu^4-30\mu^2+3}{8}) + \cdots\right\} \\
&= \frac{V}{\sqrt{2}}\frac{a}{r} - \frac{V}{4}\left\{\left(\frac{a}{r}\right)^3 \frac{5}{\sqrt{2}}\left(\frac{3\mu^2-1}{2}\right) + \left(\frac{a}{r}\right)^5 \frac{9}{4\sqrt{2}}\left(\frac{7}{2}-3\right)\frac{(35\mu^4-30\mu^2+3}{8}) + \cdots\right\} \\
\frac{\Phi(r,\theta)}{V} &= \frac{1}{\sqrt{2}}\frac{a}{r} - \frac{1}{8\sqrt{2}}\left\{5\left(\frac{a}{r}\right)^3(3\mu^2-1) + \left(\frac{a}{r}\right)^5 \frac{9}{32}(35\mu^4-30\mu^2+3) + \cdots\right\}
\end{aligned}$$



Black: $\theta = 0 \quad \mu = 1$

Red $\theta = \pi/6$ $\mu = \sqrt{3}/2$

Blue $\theta = \frac{\pi}{4}$ $\mu = \frac{1}{\sqrt{2}}$

Green $\theta = \frac{\pi}{6} \quad \mu = \frac{1}{2}$

Notice how spherical symmetry emerges for $r > 4a$. The series converges slowly as r approaches a, so we need more terms to get accurate results there.

1.6 Cone- region.

If our volume of interest is the interior of a cone with opening angle α, we no longer have $f(\mu) = 1 - \mu^2 = 0$ at the boundary $\theta = \alpha$ to give us orthogonality, so we need a boundary condition at $\mu = \cos\alpha$. For example, a grounded surface requires

$$P_\nu(\cos\alpha) = 0$$

This gives a set of eigenvalues ν. See J Fig 3.6

For example,

$$\begin{aligned} P_2(\mu) &= \frac{1}{2}\left(3\mu^2 - 1\right) = 0 \text{ for } \mu = 1/\sqrt{3} \\ \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) &= 0.955\,32 \text{ radians} = 54.736 \text{ degrees} \end{aligned}$$

So if $\alpha = 0.955$ radians, then one of the eigenvalues is $\nu = 2$.

The potential has the form

$$\Phi = \sum_\nu a_v r^\nu P_v(\mu)$$

which is finite at the origin. Near the origin, the lowest value of ν dominates:

$$E \sim r^{\nu_{\min} - 1}$$

So $E \to 0$ as $r \to 0$ for $\nu_{\min} > 1$ or $\alpha < 90^\circ$. This is the expected result. The electric field is small in a hole amd large near a spike.

1.7 Solution without azimuthal symmetry.

When a problem does not have rotational symmetry about the polar axis we need a set of eigenfunctions for which the separation constant m has non-zero values. Then the equation for the $\theta-$function is equation (2), where we keep the value $k = l(l+1)$ for that separation constant. The equation is of Sturm-Liouville form with $f(\mu) = 1 - \mu^2$, $g(\mu) = m^2/\left(1-\mu^2\right)$, $w(\mu) = 1$ and $\lambda = l(l+1)$. (Note that m is the eigenvalue for the ϕ equation.)

The solutions of this equation are the Associated Legendre functions $P_l^m(\mu)$. They satisfy the orthogonality relation:

$$\int_{-1}^{+1} P_l^m(\mu) P_{l'}^m d\mu = 0 \text{ unless } l = l'$$

where the value of m is the same in both functions. In Lea, we show that the form of the solution is

$$P_l^m(\mu) = (-1)^m \left(1-\mu^2\right)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu) \tag{19}$$

Clearly $P_l^m = 0$ for $m > l$, since the highest power of μ that appears in P_l is μ^l. Also, since the associated Legendre equation contains m^2, the eigenvalue $-m$ leads to the same differential equation. It is convenient to define

$$P_l^{-m}(\mu) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\mu) \tag{20}$$

as the appropriate solution corresponding to the eigenvalue $-m$. (This gives the second solution for the function $W(\phi)$.)

The orthogonality integral is:

$$\int_{-1}^{+1} P_l^m(\mu) P_{l'}^m d\mu = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'} \tag{21}$$

1.7.1 Spherical harmonics

The general solution to Laplace's equation in spherical coordinates may then be written as:

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left(a_{lm} r^l + \frac{b_{lm}}{r^{l+1}} \right) P_l^m(\mu) e^{im\phi}$$

Next we define the combination

$$\sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\mu) e^{im\phi} \equiv Y_{lm}(\theta, \phi) \tag{22}$$

where the constant has been chosen to make the functions Y_{lm} *orthonormal*, that is:

$$\int_{-1}^{+1} \int_0^{2\pi} Y_{lm}(\theta,\phi) Y_{l'm'}^*(\theta,\phi) d\phi d\mu = \delta_{ll'} \delta_{mm'} = \int_{\text{sphere}} Y_{lm}(\theta,\phi) Y_{l'm'}^*(\theta,\phi) d\Omega \tag{23}$$

The functions $Y_{lm}(\theta,\phi)$ are called *spherical harmonics.* They find application not only in potential problems, but in the quantum mechanics of atoms, wave mechanics, and oscillations of spheres (for example, the sun.)

With P_l^{-m} defined as in eqn (20), we have the nice result

$$Y_{l,-m} = (-1)^m Y_{lm}^* \tag{24}$$

1.7.2 Addition theorem

We may express $P_l(\cos\gamma)$ (in eqn 15) in terms of spherical harmonics (see Lea pg 390, Jackson §3.6):

$$P_l(\cos\gamma) = \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta'\phi')$$

Then from (15), we get

$$\begin{aligned} \frac{1}{|\vec{x} - \vec{x}'|} &= \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta'\phi') \end{aligned} \tag{25}$$

This is result is very useful when using expression (29) in notes 1 to find the potential from the charge density.

1.7.3 Problem:

Two concentric rings of charge have radii a and, b, equal line charge density λ, and are oriented at right angles. Find the potential everywhere.

Choose spherical coordinates with origin at the center of both rings, with polar axis along the axis of one and a diameter of the other Then the charge density due to the vertical ring is

$$\rho_a(\vec{x}) = A\lambda\delta(r-a)\left[\delta(\phi)+\delta(\phi-\pi)\right]$$

We find A by calculating the charge on a differential piece[3] of the ring:

$$\begin{aligned} dq &= 2\lambda a d\theta = \int_0^{2\pi}\int_0^{\infty}\rho_a(\vec{x})\, r^2 dr d\mu d\phi = \int_0^{2\pi}\int_0^{\infty} A\lambda\delta(r-a)\left[\delta(\phi)+\delta(\phi-\pi)\right] r^2 dr d\mu d\phi \\ &= 2A\lambda a^2\sin\theta d\theta \end{aligned}$$

Thus

$$A = \frac{1}{a\sin\theta}$$

$$\rho_a(\vec{x}) = \frac{\lambda}{a\sin\theta}\delta(r-a)\left[\delta(\phi)+\delta(\phi-\pi)\right]$$

For the horizontal ring:

$$\rho_b(\vec{x}) = B\lambda\delta(r-b)\,\delta(\mu)$$

where the charge on a differential piece of this ring is

$$\begin{aligned} dq &= \lambda b d\phi = \int_{-1}^{+1}\int_0^{\infty} B\lambda\delta(r-b)\,\delta(\mu)\, r^2 dr d\mu d\phi \\ &= B\lambda b^2 d\phi \end{aligned}$$

So

$$B = \frac{1}{b}$$

and

$$\rho_b(\vec{x}) = \frac{\lambda}{b}\delta(r-b)\,\delta(\mu)$$

[3] At any θ, there are actually two differential pieces, one on each side of the ring.

Now we compute the potential, also in two parts.

$$\begin{aligned}
\Phi_a(\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_a(\vec{x}')}{|\vec{x}-\vec{x}'|}dV' \\
&= \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\lambda}{a\sin\theta'}\frac{\delta(r'-a)\left[\delta(\phi')+\delta(\phi'-\pi)\right]}{|\vec{x}-\vec{x}'|}dV' \\
&= \frac{\lambda}{4\pi\varepsilon_0 a}\int_0^{2\pi}\int_{-1}^{+1}\int_0^{\infty}\frac{1}{\sin\theta'}\delta(r'-a)\left[\delta(\phi')+\delta(\phi'-\pi)\right] \\
&\qquad \times\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta'\phi')\,(r')^2\,dr'd\mu'd\phi' \\
&= \frac{\lambda}{\varepsilon_0 a}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{a^2}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\int\frac{N_{lm}}{\sin\theta'}P_l^m(\mu')\left(1+e^{-im\pi}\right)d\mu'
\end{aligned}$$

where $r_< = \min(r,a)$ and $N_{lm}=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}$.

Since $e^{-im\pi}=(-1)^m$, only even m survive, (as expected from reflection symmetry about $\phi=\pi/2$) and then

$$\Phi_a=\frac{2\lambda a}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l,\text{ even}}^{+l}\frac{N_{lm}}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\int_0^{\pi}P_l^m(\mu')\,d\theta'$$

Separating out the first few terms, we have

$l=0$, $m=0$

$$\Phi_{a,00}=\frac{2\lambda a}{\varepsilon_0}\frac{1}{4\pi}\frac{1}{r_>}\pi=\frac{\lambda a}{2\varepsilon_0}\frac{1}{r_>}$$

$l=l$, $m=0$

$$\Phi_{a,l0}=\frac{2\lambda a}{\varepsilon_0}\frac{1}{4\pi}\frac{r_<^l}{r_>^{l+1}}P_l(\mu)\int_{-1}^{1}\frac{P_l(\mu')}{\sqrt{1-(\mu')^2}}d\mu'$$

The integrand is odd if l is odd, and so the integral is zero. Thus only even l survive, as expected from reflection symmetry about $\mu=0$. The integral is in Lea Problem 8.8:

$$\int_{-1}^{1}\frac{P_{2n}(\mu')}{\sqrt{1-(\mu')^2}}d\mu'=\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\pi$$

$$\begin{aligned}
\Phi_a &= \frac{\lambda a}{\pi\varepsilon_0}\left\{\frac{\pi}{2r_>}+\frac{\pi}{2}\sum_{l=2,\text{even}}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\mu)\left[\frac{(l-1)!!}{l!!}\right]^2\right. \\
&\qquad\left.+\sum_{l=2,\text{even}}^{\infty}\sum_{m=2\text{ even}}^{+l}\frac{(l-m)!}{(l+m)!}\frac{r_<^l}{r_>^{l+1}}P_l^m(\mu)\cos m\phi\int_0^{\pi}P_l^m(\mu')\,d\theta'\right\}
\end{aligned}$$

Similarly for Φ_b

$$\begin{aligned}
\Phi_b &= \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\lambda}{b}\delta\left(r'-b\right)\delta\left(\mu'\right)\frac{1}{\left|\vec{x}-\vec{x}'\right|}\left(r'\right)^2 dr' d\mu' d\phi' \\
&= \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\lambda}{b}\delta\left(r'-b\right)\delta\left(\mu'\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}\left(\theta,\phi\right)Y_{lm}^*\left(\theta'\phi'\right)\left(r'\right)^2 dr' d\mu' d\phi' \\
&= \frac{\lambda b}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{Y_{lm}\left(\theta,\phi\right)}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int_0^{2\pi} Y_{lm}^*\left(\pi/2,\phi'\right)d\phi'
\end{aligned}$$

where now $r_<$ is the smaller of r and b. Only $m=0$ survives the integration over ϕ, so

$$\begin{aligned}
\Phi_b &= \frac{2\pi\lambda b}{\varepsilon_0}\sum_{l=0}^{\infty}\frac{N_{l0}^2 P_l\left(\mu\right)P_l\left(0\right)}{2l+1}\frac{r_<^l}{r_>^{l+1}} \\
&= \frac{\lambda b}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l\left(\mu\right)P_l\left(0\right)
\end{aligned}$$

where $r_< = \min\left(r,b\right)$ and only even values of l have $P_l\left(0\right)\neq 0$. Again this indicates the reflection symmetry about the $\mu=0$ plane.

Thus for $r<a<b$

$$\begin{aligned}
\Phi &= \frac{\lambda}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{b^l}P_l\left(\mu\right)P_l\left(0\right)+\frac{\lambda}{2\varepsilon_0}\left\{1+\sum_{l=2,\text{even}}^{\infty}\frac{r^l}{a^l}P_l\left(\mu\right)\left[\frac{(l-1)!!}{l!!}\right]^2\right\} \\
&\quad +\frac{\lambda}{\pi\varepsilon_0}\left[\sum_{l=2,\text{even}}^{\infty}\sum_{m=2\text{ even}}^{+l}\frac{(l-m)!}{(l+m)!}\frac{r^l}{a^l}P_l^m\left(\mu\right)\cos m\phi\int_0^{\pi}P_l^m\left(\mu'\right)d\theta'\right]
\end{aligned}$$

The potential at $r=0$ is

$$\Phi\left(0\right)=\frac{\lambda}{\varepsilon_0}=\frac{2\pi\lambda a}{4\pi\varepsilon_0 a}+\frac{2\pi\lambda b}{4\pi\varepsilon_0 b}=\frac{Q_a}{4\pi\varepsilon_0 a}+\frac{Q_b}{4\pi\varepsilon_0 b}$$

as expected, since all the charge on each ring is at the same distance from the origin..

For $a<r<b$ we get

$$\begin{aligned}
\Phi &= \frac{\lambda}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{b^l}P_l\left(\mu\right)P_l\left(0\right)+\frac{\lambda}{2\varepsilon_0}\left\{\frac{a}{r}+\sum_{l=2,\text{even}}^{\infty}\frac{a^{l+1}}{r^{l+1}}P_l\left(\mu\right)\left[\frac{(l-1)!!}{l!!}\right]^2\right\} \\
&\quad +\frac{\lambda}{\pi\varepsilon_0}\left[\sum_{l=2,\text{even}}^{\infty}\sum_{m=2\text{ even}}^{+l}\frac{(l-m)!}{(l+m)!}\frac{a^{l+1}}{r^{l+1}}P_l^m\left(\mu\right)\cos m\phi\int_0^{\pi}P_l^m\left(\mu'\right)d\theta'\right]
\end{aligned}$$

while for $a < b < r$

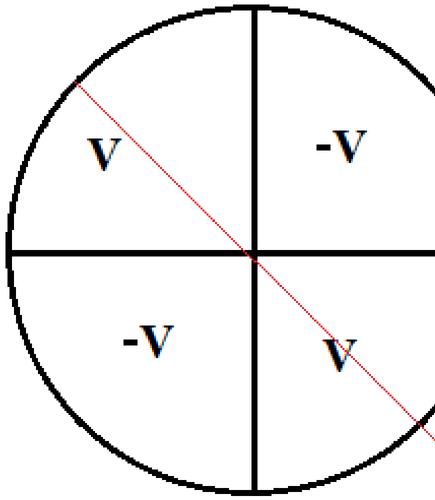
$$\begin{aligned}\Phi &= \frac{\lambda}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{b^{l+1}}{r^{l+1}}P_l(\mu)P_l(0)+\frac{\lambda}{2\varepsilon_0}\left\{\frac{a}{r}+\sum_{l=2,\text{even}}^{\infty}\frac{a^{l+1}}{r^{l+1}}P_l(\mu)\left[\frac{(l-1)!!}{l!!}\right]^2\right\}\\ &\quad+\frac{\lambda}{\pi\varepsilon_0}\left[\sum_{l=2,\text{even}}^{\infty}\sum_{m=2\text{ even}}^{+l}\frac{(l-m)!}{(l+m)!}\frac{a^{l+1}}{r^{l+1}}P_l^m(\mu)\cos m\phi\int_0^{\pi}P_l^m(\mu')\,d\theta'\right]\end{aligned}$$

At very great distances $r \gg b$, the $l = 0$ term dominates and

$$\Phi \simeq \frac{\lambda}{2\varepsilon_0}\frac{b}{r}+\frac{\lambda a}{2\varepsilon_0 r}=\frac{2\pi\lambda(a+b)}{4\pi\varepsilon_0 r}=\frac{Q_a+Q_b}{4\pi\varepsilon_0 r}$$

as expected.

Potential on surface of sphere is $\pm V$ on alternate quarters.



Using Y_{lm}

$$\Phi = \sum_{l,m} A_{lm} r^l Y_{lm}(\theta,\phi)$$

$$\sum_{l,m} A_{lm} a^l Y_{lm}(\theta,\phi) = \begin{array}{l} -V \text{ if } 0<\phi<\pi \text{ and } 1\geq\mu>0 \text{ OR } \pi<\phi<2\pi \text{ and } 0>\mu\geq -1 \\ V \text{ if } 0\leq\phi\leq\pi \text{ and } 0>\mu\geq -1 \text{ OR } \pi<\phi<2\pi \text{ and } 1\geq\mu>0 \end{array}$$

By orthogonality of the Y_{lm}, we have

$$\begin{aligned} \int_{\text{sphere}} \Phi(a,\theta,\phi) Y^*_{l'm'}(\theta,\phi)\, d\Omega &= \sum_{l,m} A_{lm} a^l \int_{\text{sphere}} Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi)\, d\Omega \\ &= \sum_{l,m} A_{lm} a^l \delta_{ll'}\delta_{mm'} = A_{l'm'} a^{l'} \end{aligned}$$

Thus

$$A_{lm}a^l = VN_{lm}\left\{\int_0^{+1} P_l^m(\mu)\,d\mu\left(-\int_0^{\pi}+\int_{\pi}^{2\pi}\right)e^{-im\phi}d\phi \int_{-1}^{0} P_l^m(\mu)\,d\mu\left(\int_0^{\pi}-\int_{\pi}^{2\pi}\right)e^{-im\phi}d\phi\right\}$$

If $m=0$, the ϕ integral is zero. So for $m\neq 0$, we get

$$\begin{aligned} A_{lm}a^l &= VN_{lm}\frac{1}{-im}\left\{\int_0^{+1} P_l^m(\mu)\,d\mu\left(-(-1)^m+1+1-(-1)^m\right)+\int_{-1}^{0} P_l^m(\mu)\,d\mu\left((-1)^m-1-1+(-1)^m\right)\right\} \\ &= -VN_{lm}\frac{4}{im}\left\{\int_0^{+1} P_l^m(\mu)\,d\mu-\int_{-1}^{0} P_l^m(\mu)\,d\mu\right\} \quad \text{if } m \text{ is odd and zero otherwise.} \\ &= -VN_{lm}\frac{4}{im}\left\{\int_0^{+1} P_l^m(\mu)\,d\mu-\int_{1}^{0} P_l^m(-\mu)\,d(-\mu)\right\} = -VN_{lm}\frac{4}{im}\left\{\int_0^{+1}\left[P_l^m(\mu)-P_l^m(-\mu)\right]d\mu\right\} \end{aligned}$$

The result is zero unless m is odd, as we would expect from the reflection anti-symmetry about the lines $\phi = 0, \pi$ We also need $l - m$ to be odd,which is expected from the reflection anti-symmetry about the plane $\mu = 0$, so l must be even. . Then

$$A_{lm} = -VN_{lm}\frac{8}{im}\left\{\int_0^{+1} P_l^m(\mu)\,d\mu\right\}$$

Label the integral

$$I_{lm} = \int_0^{+1} P_l^m(\mu)\,d\mu$$

The potential inside the sphere is

$$\Phi(r,\theta,\phi) = -V\sum_{l=2,\text{even}}^{\infty}\sum_{m=-l,\text{odd}}^{l} N_{lm}^2 I_{lm}\left(\frac{r}{a}\right)^l \frac{8}{m}P_l^m(\mu)\frac{e^{im\phi}}{i}$$

To show that the result is real, we combine the positive and negative m terms.

$$\begin{aligned} Y_{l,-m} &= (-1)^m Y_{lm}^* \\ N_{l,-m}P_l^{-m}e^{-im\phi} &= (-1)^m N_{lm}P_l^m e^{-im\phi} \end{aligned}$$

So, for odd m, we get

$$N_{l,-m}P_l^{-m} = (-1)^m N_{lm}P_l^m = -N_{lm}P_l^m$$

and thus

$$N_{l,-m}I_{l,-m} = -N_{lm}I_{lm}$$

thus

$$\begin{aligned} \Phi(r,\theta,\phi) &= -V\sum_{l=2,\text{even}}^{\infty}\sum_{m=1,\text{odd}}^{l} N_{lm}^2 I_{lm}\left(\frac{r}{a}\right)^l \frac{8}{m}P_l^m(\mu)\frac{e^{im\phi}-e^{-im\phi}}{i} \\ &= -16V\sum_{l=2,\text{even}}^{\infty}\sum_{m=1,\text{odd}}^{l} N_{lm}^2 I_{lm}\left(\frac{r}{a}\right)^l P_l^m(\mu)\frac{\sin m\phi}{m} \\ &= -16V\sum_{l=2,\text{even}}^{\infty}\sum_{m=1,\text{odd}}^{l} \frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!} I_{lm}\left(\frac{r}{a}\right)^l P_l^m(\mu)\frac{\sin m\phi}{m} \end{aligned}$$

The first few terms are

$$\begin{aligned} \Phi(r,\theta,\phi) &= -\frac{4V}{\pi}\left[\frac{5}{3!}I_{21}\left(\frac{r}{a}\right)^2 P_2^1(\mu)\sin\phi + 9\left(\frac{r}{a}\right)^4\left(\frac{3!}{5!}I_{41}P_4^1(\mu)\sin\phi + \frac{1}{7!}I_{43}P_4^3(\mu)\frac{\sin 3\phi}{3}\right) + \cdots\right] \\ &= -\frac{2V}{\pi}\left[\frac{5}{3}I_{21}\frac{r}{a}P_2^1(\mu)\sin\phi + 9\left(\frac{r}{a}\right)^4\left(\frac{1}{10}I_{41}P_4^1(\mu)\sin\phi + \frac{1}{7560}I_{43}P_4^3(\mu)\sin 3\phi\right)\right] \end{aligned}$$

where

$$\begin{aligned} P_2^1 &= -3\cos\theta\sin\theta \\ I_{21} &= -\int_0^{\pi/2} 3\cos\theta\sin^2\theta\,d\theta = -\sin^3\theta\Big|_0^{\pi/2} \\ &= -1 \end{aligned}$$

$$
\begin{aligned}
P_4^1(\mu) &= \frac{5}{2}\cos\theta\left(3-7\cos^2\theta\right)\sin\theta \\
I_{41} &= \int_0^{\pi/2}\frac{5}{2}\cos\theta\left(3-7\cos^2\theta\right)\sin^2\theta d\theta \\
&= \frac{5}{2}\int_0^{\pi/2}\cos\theta\left(-4+7\sin^2\theta\right)\sin^2\theta d\theta = \frac{5}{2}\left(-4\frac{\sin^3\theta}{3}+\frac{7\sin^5\theta}{5}\right)\Bigg|_0^{\pi/2} \\
&= \frac{5}{2}\left(-\frac{4}{3}+\frac{7}{5}\right) = \frac{1}{6}
\end{aligned}
$$

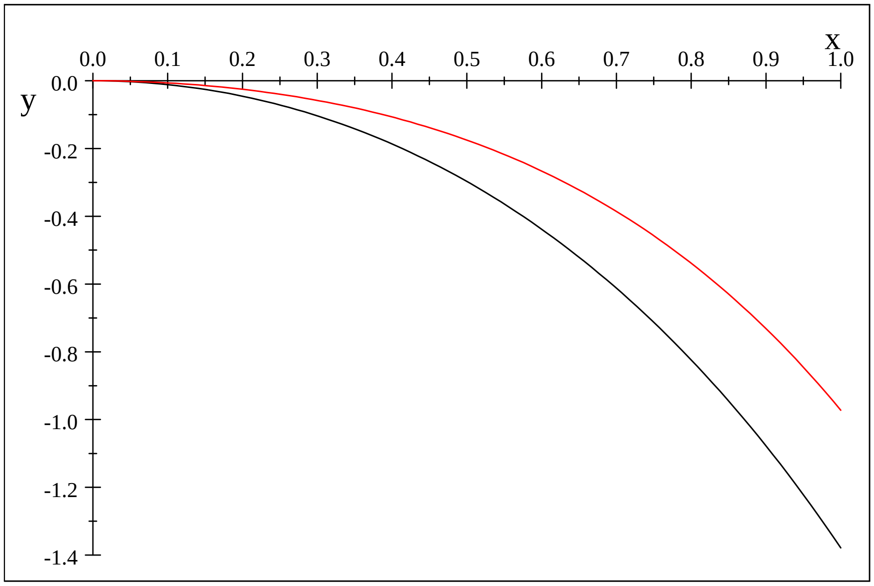
$$
\begin{aligned}
P_4^3(\mu) &= -105\cos\theta\sin^3\theta \\
I_{43} &= -105\int_0^{\pi/2}\cos\theta\sin^4\theta d\theta = -105\left(\frac{\sin^5\theta}{5}\right)\Bigg|_0^{\pi/2} = -21
\end{aligned}
$$

So

$$
\begin{aligned}
\Phi(r,\theta,\phi) &= -\frac{2V}{\pi}\left[\frac{5}{3}\frac{r}{a}3\cos\theta\sin\theta\sin\phi+9\left(\frac{r}{a}\right)^4\left(\frac{1}{10}\frac{1}{6}\frac{5}{2}\cos\theta\left(3-7\cos^2\theta\right)\sin\theta\sin\phi+\frac{1}{7560}21\times 105\cos\theta\sin^3\theta\sin 3\phi\right)\right] \\
&= -\frac{2V}{\pi}\left[5\frac{r}{a}\cos\theta\sin\theta\sin\phi+9\left(\frac{r}{a}\right)^4\left(\frac{1}{24}\cos\theta\left(3-7\cos^2\theta\right)\sin\theta\sin\phi+\frac{7}{24}\cos\theta\sin^3\theta\sin 3\phi\right)\right] \\
&= -\frac{2V}{\pi}\left(\frac{r}{a}\right)^2\sin\theta\cos\theta\left[5\sin\phi+9\left(\frac{r}{a}\right)^4\left(\frac{1}{24}\left(3-7\cos^2\theta\right)\sin\phi+\frac{7}{24}\sin^2\theta\sin 3\phi\right)\right] \\
&= -\frac{V}{\pi}\left(\frac{r}{a}\right)^2\sin 2\theta\left[5\sin\phi+9\left(\frac{r}{a}\right)^2\left(\frac{1}{24}\left(3-7\cos^2\theta\right)\sin\phi+\frac{7}{24}\sin^2\theta\sin 3\phi\right)\right]
\end{aligned}
$$

The potential is zero at $\theta = 0, \frac{\pi}{2}$ as it must be on the boundary between $\pm V$, and is maximum at $\theta = \pi/4$, $\phi = \pi/4$, as expected. The potential also changes sign when $\cos\theta$ does, also as expected. It is also zero at $r = 0$, the average of the value on the surface of the sphere.

The plot shows Φ/V vs r/a at $\theta = \pi/4, \phi = \pi/4$ (black) and $\pi/8$ (red) $-\frac{1}{\pi}r^2\left(\sin\frac{\pi}{2}\right)\left(5\sin\frac{\pi}{8}+9r^2\left(\frac{1}{24}\left(3-7\cos^2\frac{\pi}{4}\right)\sin\frac{\pi}{8}+\frac{7}{24}\sin^2\frac{\pi}{4}\sin\frac{3\pi}{8}\right)\right)$



The values are less accurate near $r = a$. Why is that?
The plot shows contours of constant Φ/V at, $\phi = \pi/4$

$$\frac{\Phi}{V} = -\frac{1}{\pi}\left(\frac{r}{a}\right)^2 \sin 2\theta \left[5\sin\phi + 9\left(\frac{r}{a}\right)^2\left(\frac{1}{24}\left(3 - 7\cos^2\theta\right)\sin\phi + \frac{7}{24}\sin^2\theta\sin 3\phi\right)\right]$$

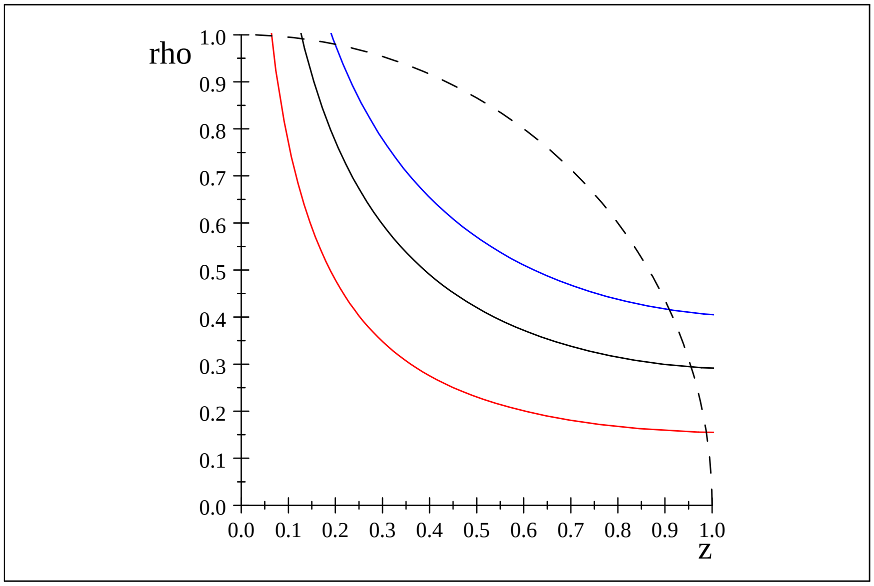
Let $u = \left(\frac{r}{a}\right)^2$. Then the equipotentials are given by:

$$5u\sin\phi + 9u^2\left(\frac{1}{24}\left(3 - 7\cos^2\theta\right)\sin\phi + \frac{7}{24}\sin^2\theta\sin 3\phi\right) + \frac{C\pi}{\sin 2\theta} = 0$$

$$u = \frac{-5\sin\phi \pm \sqrt{25\sin^2\phi - 4\frac{C\pi}{\sin 2\theta} \times 9\left(\frac{1}{24}\left(3 - 7\cos^2\theta\right)\sin\phi + \frac{7}{24}\sin^2\theta\sin 3\phi\right)}}{18\left(\frac{1}{24}\left(3 - 7\cos^2\theta\right)\sin\phi + \frac{7}{24}\sin^2\theta\sin 3\phi\right)}$$

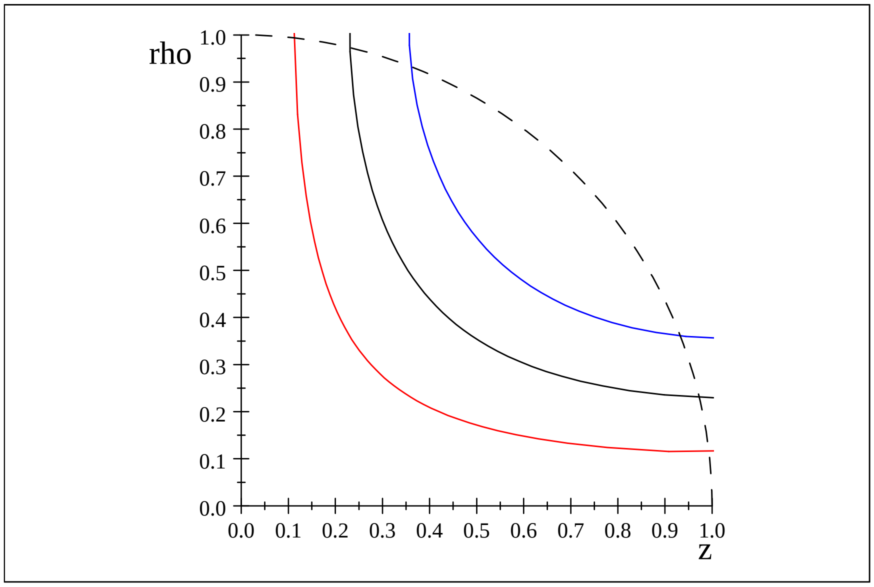
Only the + sign makes sense, so at $\phi = \pi/4$ we have

$$\frac{r}{a} = \sqrt{\frac{-5\sin\frac{\pi}{4} + \sqrt{25\sin^2\frac{\pi}{4} - 36\frac{C\pi}{\sin 2\theta}\left(\frac{1}{24}\left(3 - 7\cos^2\theta\right)\sin\frac{\pi}{4} + \frac{7}{24}\sin^2\theta\sin 3\frac{\pi}{4}\right)}}{18\left(\frac{1}{24}\left(3 - 7\cos^2\theta\right)\sin\frac{\pi}{4} + \frac{7}{24}\sin^2\theta\sin\frac{3\pi}{4}\right)}}$$



Values of Φ/V are: Blue -3/4, black -1/2, Red-1/4
At $\phi = \pi/2$,

$$\frac{r}{a} = \sqrt{\frac{-5+\sqrt{25-36\frac{C\pi}{\sin 2\theta}\left(\frac{1}{24}\left(3-7\cos^2\theta\right)-\frac{7}{24}\sin^2\theta\right)}}{18\left(\frac{1}{24}\left(3-7\cos^2\theta\right)-\frac{7}{24}\sin^2\theta\right)}}$$



Note: to get azimuthal symmetry, we put the polar axis as shown by the red line in the diagram. Then at $r = a$:

$$\begin{aligned}\Phi &= +V \text{ if } 0 \leq \theta < \frac{\pi}{4} \text{ or } 3\frac{\pi}{4} < \theta \leq \pi \\ &= -V \text{ if } \frac{\pi}{4} < \theta < \frac{3\pi}{4}\end{aligned}$$

and we can write tthe potential as

$$\Phi = \sum_{l=0}^{\infty} A_l r^l P_l(\mu)$$

where

$$\begin{aligned}A_l a^l \frac{2}{2l+1} &= V\left\{\int_{1/\sqrt{2}}^{1} + \int_{-1}^{-1/\sqrt{2}} - \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\right\} P_l(\mu)\, d\mu \\ &= \frac{V}{2l+1} \left(P_{l+1}(\mu) - P_{l-1}(\mu)\right)\Big|_{1/\sqrt{2}}^{1} + \Big|_{-1}^{-1/\sqrt{2}} - \Big|_{-1/\sqrt{2}}^{+1/\sqrt{2}} \\ A_l a^l &= \frac{V}{2}\left[P_{l+1}(1) - P_{l+1}\left(\frac{1}{\sqrt{2}}\right) + P_{l+1}\left(-\frac{1}{\sqrt{2}}\right) - P_{l+1}(-1) - P_{l+1}\left(\frac{1}{\sqrt{2}}\right) + P_{l+1}\left(-\frac{1}{\sqrt{2}}\right) - (l+1 \to l-1)\right] \\ &= V\left[-P_{l+1}\left(\frac{1}{\sqrt{2}}\right) + P_{l+1}\left(-\frac{1}{\sqrt{2}}\right) + P_{l-1}\left(\frac{1}{\sqrt{2}}\right) - P_{l-1}\left(-\frac{1}{\sqrt{2}}\right)\right] \quad l > 0\end{aligned}$$

The result is zero if $l - 1$ is even (l odd) and for l even we get

$$A_l = 2\frac{V}{a^l}\left[P_{l-1}\left(\frac{1}{\sqrt{2}}\right) - P_{l+1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

For $l = 0$ we have

$$\begin{aligned}2A_0 &= V\left\{\int_{1/\sqrt{2}}^{1} + \int_{-1}^{-1/\sqrt{2}} - \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\right\} d\mu = V\left(1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \\ &= 2V\left(1 - \sqrt{2}\right)\end{aligned}$$

Should be zero!

So

$$\begin{aligned}\Phi &= 2V \sum_{l=0,\text{even}}^{\infty} \left[P_{l-1}\left(\frac{1}{\sqrt{2}}\right) - P_{l+1}\left(\frac{1}{\sqrt{2}}\right)\right] \left(\frac{r}{a}\right)^l P_l(\mu) \\ &= 2V \sum_{l=0,\text{even}}^{\infty} \left[\frac{1 - 1/2}{l} P_l'\left(\frac{1}{\sqrt{2}}\right) + \frac{1 - 1/2}{l+1} P_l'\left(\frac{1}{\sqrt{2}}\right)\right] \left(\frac{r}{a}\right)^l P_l(\mu) \\ &= V \sum_{l=0,\text{ even}}^{\infty} \frac{2l+1}{l(l+1)} P_l'\left(\frac{1}{\sqrt{2}}\right) \left(\frac{r}{a}\right)^l P_l(\mu)\end{aligned}$$

The first few terms are

$$\Phi = V \sum_{l=0,\text{even}}^{\infty} \frac{2l+1}{l\,(l+1)} P_l' \left(\frac{1}{\sqrt{2}} \right) \left(\frac{r}{a} \right)^l P_l\,(\mu)$$

$$-\frac{V}{2}\frac{r}{a}\sin\theta \left[3\sin\phi + \frac{7}{16}\left(\frac{r}{a}\right)^2 \left(3\left(4-5\sin^2\theta\right)\sin\phi + 5\sin^2\theta\sin 3\phi\right)\right]$$