Jackosn notes 2020

Sturm-Liouville Review:

Many differential equations describing physical systems can be reduced to one or more linear ordinary differential equations of the form

$$\frac{d}{dx}\left(f\left(x\right)\frac{dy}{dx}\right)-g\left(x\right)y+\lambda w\left(x\right)y=0 \tag{1}$$

where $w\left(x\right)\geq 0$, and the solution $y\left(x\right)$ also obeys boundary conditions of the form

$$\alpha_1 y+\beta_1\frac{dy}{dx}=0 \text{ at } x=a \tag{2}$$

and

$$\alpha_2 y+\beta_2\frac{dy}{dx}=0 \text{ at } x=b$$

We want to determine values of the constant λ for which there are non-trivial solutions $y\left(x\right)$. This is called the Sturm-Liouville problem. If $\alpha=0$ the boundary condition simplifies to $dy/dx=0$ (Neumann conditions) or if $\beta=0$ the condition is $y=0$ (Dirichlet conditions). The constants α and β cannot both be zero.

Orthogonality of the eigenfunctions

$$\int_a^b w\left(x\right)y_m y_n dx=0 \quad n\neq m \tag{3}$$

We also obtain the orthogonality integral for complex functions:

$$\int_a^b w\left(x\right)y_n^* y_m dx=0, \qquad n\neq m \tag{4}$$

There are two other important cases which lead to orthogonal functions. If the function $f\ \left(x\right)$ in (1) has the value zero at $x=a$ and $x=b$, then the integrated term is zero no matter what the boundary conditions on $y\left(x\right)$. We shall see that this is the case for Legendre's equation. Finally, if $yy'f$ has period $\left(b-a\right)$ then (3) or (4) results.

When equation (3) or eqn (4) is satisfied, the set of eigenfunctions $y_n\left(x\right)$ form a complete orthogonal set on the interval $\left[a,b\right]$. This means that any reasonably well-behaved function $f\left(x\right)$ defined for $a\leq x\leq b$ can be expanded in a series of eigenfunctions that is weakly convergent:

$$f\left(x\right)=\sum_{n=0}^{\infty}a_n y_n\left(x\right) \tag{5}$$

where the coefficients a_n may be found, as for Fourier series, by multiplying both sides of relation (5) by $w\left(x\right)y_m\left(x\right)$ and integrating over the range a to b.

The sum and integral of a weakly convergent series may be interchanged. Only the one term in the sum with $m = n$ survives the integration, and

$$a_m = \frac{\int_a^b f(x)\, y_m(x)\, w(x)\, dx}{\int_a^b [y_m(x)]^2\, w(x)\, dx}$$

As we found with the Fourier series in Chapter 4, the sum converges to the function in the mean, that is:

$$\lim_{N\to\infty} \int_a^b \left[f(x) - \sum_{n=0}^{N} a_n y_n(x) \right]^2 w(x)\, dx = 0$$