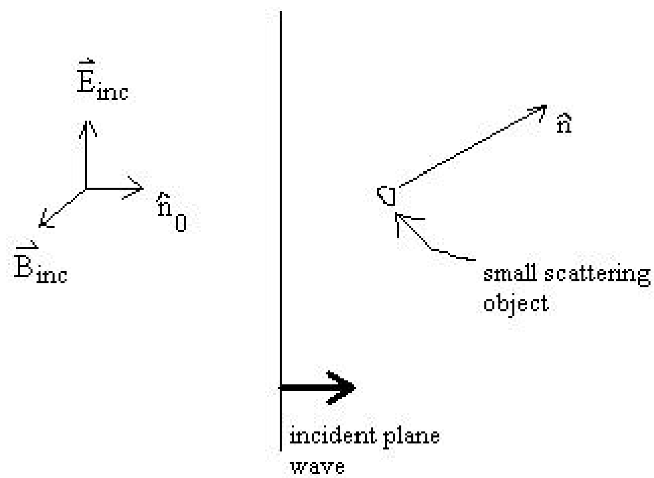
# 1 Scattering in the long wavelength limit

Suppose a plane wave with $\vec{E}_{\rm inc} = \vec{\varepsilon}_0 E_0 e^{ik\hat{n}_0\cdot\vec{x}}$ is incident on a small scattering object with dimension $d \ll \lambda$. The incoming wave induces electric and magnetic dipole monents in the scattering object, which then radiates.




The scattered fields are

$$\vec{E}_{\rm sc} = \frac{k^2 e^{ikr}}{r}\left[(\hat{n}\times\vec{p})\times\hat{n} - \hat{n}\times\vec{m}\right]$$

and

$$\vec{B}_{\rm sc} = \hat{n}\times\vec{E}_{\rm sc}$$

The power radiated into direction $\hat{n}$ with polarization $\vec{\varepsilon}$ per unit solid angle is

$$\frac{dP}{d} = r^2\frac{c}{4\pi}\left|\vec{\varepsilon}^*\cdot\vec{E}_{\rm sc}\right|^2$$

and the differential scattering cross section is

$$\begin{aligned}
\frac{d\sigma}{d}\left(\hat{n},\vec{\varepsilon},\hat{n}_0,\vec{\varepsilon}_0\right) &= \frac{r^2\frac{c}{4\pi}\left|\vec{\varepsilon}^*\cdot\vec{E}_{\rm sc}\right|^2}{\frac{c}{8\pi}\left|\vec{\varepsilon}_0^*\cdot\vec{E}_{\rm inc}\right|^2} \\
&= \frac{k^4}{E_0^2}\left|\vec{\varepsilon}^*\cdot\left[(\hat{n}\times\vec{p})\times\hat{n} - \hat{n}\times\vec{m}\right]\right|^2 \\
&= \frac{k^4}{E_0^2}\left|\vec{\varepsilon}^*\cdot\vec{p} + (\hat{n}\times\vec{\varepsilon}^*)\cdot\vec{m}\right|^2 \qquad (1)
\end{aligned}$$

where we used the fact that $\vec{\varepsilon}^* \cdot \hat{n} = 0$, and we rearranged the triple scalar product on the right.

The result shows that the differential scattering cross section is proportional to $\lambda^4$. This is Rayleigh's law. It applies to all scattering in the long-wavelength limit.

**Example**: scattering by a small conducting sphere, radius $a$.

Since the sphere is small ($a \ll \lambda$) it sees the incident field as slowly varying. The sphere can adjust to the electric field in a time $t \sim a/c \ll \lambda/c = T$, the wave period. Thus we may use the results for the static fields from chapter 3. With polar axis along $\vec{\varepsilon}_0$, the potential is

$$\phi = -E_0 \left( r - \frac{a^3}{r} \right) \cos\theta$$

The first term is the incident field and the second, dipole term is the field due to the charge distribution on the surface of the sphere. The dipole moment is $\vec{p} = \vec{E}_0 a^3$. We may also find the scalar magnetic potential. The boundary condition is $B_r = 0$ at $r = a$, giving

$$\phi_M = -B_0 \left( r + \frac{a^3}{2r} \right) \cos\theta$$

where here the polar axis is along $\vec{B}_0$, and hence

$$B_r = B_0 \left( 1 - \frac{a^3}{r^3} \right) \cos\theta$$

which is clearly zero at $r = a$. Again the first term is the incident field, and the seocnd term is a dipole field. Here the magnetic moment is

$$\vec{m} = -\frac{\vec{B}_0 a^3}{2} = -\frac{E_0 a^3}{2} \left( \hat{n}_0 \times \vec{\varepsilon}_0 \right)$$

We may now put these moments into the general result (1):

$$\begin{aligned}
\frac{d\sigma}{d} &= \frac{k^4}{E_0^2} E_0^2 a^6 \left| \vec{\varepsilon}^* \cdot \vec{\varepsilon}_0 - \frac{1}{2} \left( \hat{n} \times \vec{\varepsilon}^* \right) \cdot \left( \hat{n}_0 \times \vec{\varepsilon}_0 \right) \right|^2 \\
&= (ka)^4 a^2 \left| \vec{\varepsilon}^* \cdot \vec{\varepsilon}_0 - \frac{1}{2} \left( \hat{n} \times \vec{\varepsilon}^* \right) \cdot \left( \hat{n}_0 \times \vec{\varepsilon}_0 \right) \right|^2
\end{aligned}$$

The vectors $\hat{n}_0$ and $\hat{n}$ define the plane of scattering. $\hat{\varepsilon}$ is perpendicular to $\hat{n}$ and $\vec{\varepsilon}_0$ is perpendicular to $\hat{n}_0$. We choose polarization vectors $\vec{\varepsilon}_{0,1}$ and $\vec{\varepsilon}_1$ in the plane of scattering. Thus$\vec{\varepsilon}_1$ makes angle $\theta$ with $\vec{\varepsilon}_{0,1}$. Similarly $\vec{\varepsilon}_{0,2}$ and $\vec{\varepsilon}_2$ are perpendicular to the plane of scattering and parallel to each other. For scattering of unplarized incident radiation into polarization $\vec{\varepsilon}_1$, in the plane of scattering, we have

$$\begin{aligned}
\frac{\partial\sigma_{\parallel}}{d} &= \frac{(ka)^4 a^2}{2}\left\{\left|\vec{\varepsilon}_1^{*}\cdot\vec{\varepsilon}_{01}-\frac{1}{2}(\hat{n}\times\vec{\varepsilon}_1^{*})\cdot(\hat{n}_0\times\vec{\varepsilon}_{01})\right|^2+\left|\vec{\varepsilon}_1^{*}\cdot\vec{\varepsilon}_{02}-\frac{1}{2}(\hat{n}\times\vec{\varepsilon}_1^{*})\cdot(\hat{n}_0\times\vec{\varepsilon}_{02})\right|^2\right\} \\
&= \frac{(ka)^4 a^2}{2}\left\{\left|\cos\theta-\frac{1}{2}\right|^2+0\right\}
\end{aligned}$$

For scattering intopolarization $\vec{\varepsilon}_2$,

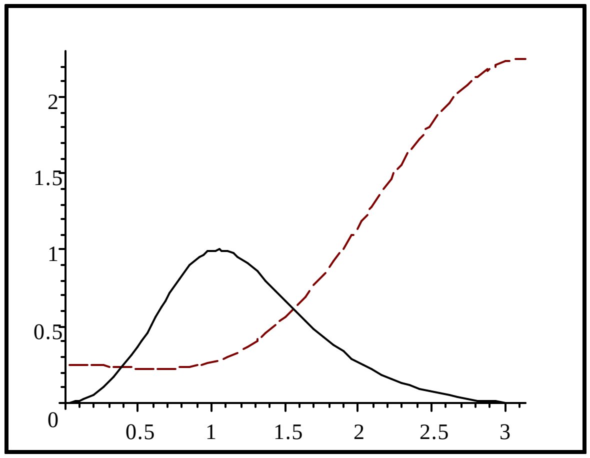
$$\begin{aligned}
\frac{\partial\sigma_{\perp}}{d} &= \frac{(ka)^4 a^2}{2}\left\{\left|\vec{\varepsilon}_2^{*}\cdot\vec{\varepsilon}_{01}-\frac{1}{2}(\hat{n}\times\vec{\varepsilon}_2^{*})\cdot(\hat{n}_0\times\vec{\varepsilon}_{01})\right|^2+\left|\vec{\varepsilon}_2^{*}\cdot\vec{\varepsilon}_{02}-\frac{1}{2}(\hat{n}\times\vec{\varepsilon}_2^{*})\cdot(\hat{n}_0\times\vec{\varepsilon}_{02})\right|^2\right\} \\
&= \frac{(ka)^4 a^2}{2}\left\{\left|0-\frac{1}{2}\vec{\varepsilon}_1^{*}\cdot\vec{\varepsilon}_{02}\right|^2+\left|1-\frac{1}{2}(-\vec{\varepsilon}_1^{*})\cdot(-\vec{\varepsilon}_{01})\right|^2\right\} \\
&= \frac{(ka)^4 a^2}{2}\left\{0+\left|1-\frac{1}{2}\cos\theta\right|^2\right\}
\end{aligned}$$

The sum of the two gives the differential scattering cross section

$$\begin{aligned}
\frac{d\sigma}{d} &= \frac{(ka)^4 a^2}{2}\left(\cos^2\theta-\cos\theta+\frac{1}{4}+1-\cos\theta+\frac{\cos^2\theta}{4}\right) \\
&= (ka)^4 a^2\left[\frac{5}{8}\left(\cos^2\theta+1\right)-\cos\theta\right]
\end{aligned}$$

The polarization is

$$\begin{aligned}
\Pi(\theta) &= \frac{\frac{d\sigma_{\perp}}{d}-\frac{\partial\sigma_{\parallel}}{d}}{\frac{d\sigma_{\perp}}{d}+\frac{\partial\sigma_{\parallel}}{d}} \\
&= \frac{\left(1-\cos\theta+\frac{\cos^2\theta}{4}\right)-\left(\cos^2\theta-\cos\theta+\frac{1}{4}\right)}{\frac{5}{4}(\cos^2\theta+1)-2\cos\theta} \\
&= \frac{\frac{3}{4}\left(1-\cos^2\theta\right)}{\frac{5}{4}(\cos^2\theta+1)-2\cos\theta} \\
&= \frac{3\sin^2\theta}{5(\cos^2\theta+1)-8\cos\theta}
\end{aligned}$$




The dashed line is the differential scattering cross section, the solid line is the polarization. The polarization peaks at $\theta = \pi/3$. The scattering peaks at $\theta = \pi$ (backward scattering) and is minimum at $\theta = \pi/5$.