Runge Kutta

We start with a first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

Then the Taylor series is:

$$\begin{aligned}
y(x_0+h) &= y_0 + hf(x_0,y_0) + \frac{h^2}{2}\frac{df}{dx} + \frac{h^3}{3!}\frac{d^2f}{dx^2}\cdots \\
&= y_0 + hf(x_0,y_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + \frac{h^3}{3!}\frac{d}{dx}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)\cdots \\
&= y_0 + hf(x_0,y_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \qquad (1)\\
&\quad + \frac{h^3}{3!}\left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 + f^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots
\end{aligned}$$

The first guess for the increment in y is the first order term in the Taylor series:

$$\eta_1 = hf(x_0,y_0) \qquad (2)$$

The second guess is

$$\eta_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{\eta_1}{2}\right)$$

First expand using a Taylor series in x, then expand in y, keeping terms up to third order in h.

$$\begin{aligned}
\eta_2 &= hf(x_0,y_0+\eta_1/2) + \frac{h^2}{2}\frac{\partial f(x_0,y_0+\eta_1/2)}{\partial x} + \frac{h^3}{4\times 2}\frac{\partial^2 f(x_0,y_0+\eta_1/2)}{\partial x^2}\cdots \\
&= hf(x_0,y_0) + h\frac{\eta_1}{2}\frac{\partial f}{\partial y} + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + \frac{\eta_1}{2}\frac{\partial^2 f}{\partial x\partial y}\right) + \frac{h^3}{8}\frac{\partial^2 f(x_0,y_0+\eta_1/2)}{\partial x^2} + \cdots \\
&= hf(x_0,y_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) + \frac{h^3}{4}f\frac{\partial^2 f}{\partial x\partial y} + \frac{h^3}{8}\frac{\partial^2 f}{\partial x^2} + \mathcal{O}\left(h^4\right) \qquad (3)
\end{aligned}$$

We used the result (2) in the last step. The third guess is

$$\begin{aligned}
\eta_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{\eta_2}{2}\right) \\
&= hf(x_0,y_0) + h\frac{\eta_2}{2}\frac{\partial f}{\partial y} + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + \frac{\eta_2}{2}\frac{\partial^2 f}{\partial x\partial y}\right) + \frac{h^3}{8}\frac{\partial^2 f(x_0,y_0+\eta_2/2)}{\partial x^2} + \cdots \\
&= hf + \frac{h^2}{2}\frac{\partial f}{\partial x} + \frac{h}{2}\frac{\partial f}{\partial y}\left[hf + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)\right] + \frac{h^2}{4}\frac{\partial^2 f}{\partial x\partial y}(hf) + \frac{h^3}{8}\frac{\partial^2 f}{\partial x^2} + \cdots \\
&= hf + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) + \frac{h^3}{4}\left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 + f\frac{\partial^2 f}{\partial x\partial y} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) + \cdots (4)
\end{aligned}$$

Again we have kept terms up to third order in h. Finally we use this result to get η_4 :

$$
\begin{aligned}
\eta_4 &= hf\left(x_0+h, y_0+\eta_3\right) \\
&= hf\left(x_0, y_0\right) + h\eta_3 \frac{\partial f}{\partial y} + h^2\left(\frac{\partial f}{\partial x} + \eta_3 \frac{\partial^2 f}{\partial x \partial y}\right) + \frac{h^3}{2} \frac{\partial^2 f\left(x_0, y_0+\eta_2/2\right)}{\partial x^2} + \cdots \\
&= hf\left(x_0, y_0\right) + h^2 \frac{\partial f}{\partial x} + h \frac{\partial f}{\partial y}\left(hf + \frac{h^2}{2} \frac{\partial f}{\partial x} + \frac{h^2}{2} f \frac{\partial f}{\partial y}\right) + h^2 \frac{\partial^2 f}{\partial x \partial y} hf + \frac{h^3}{2} \frac{\partial^2 f}{\partial x^2} + \cdots \\
&= hf + h^2 \frac{\partial f}{\partial x} + h^2 f \frac{\partial f}{\partial y} + \frac{h^3}{2}\left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 + 2f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial x^2}\right) + \cdots \qquad (5)
\end{aligned}
$$

Then the final increment is

$$
\begin{aligned}
\eta &= \frac{1}{6}\left(\eta_1 + 2\eta_2 + 2\eta_3 + \eta_4\right) \\
&= \frac{1}{6}\left[hf\left(x_0, y_0\right) + 2\left(hf\left(x_0, y_0\right) + \frac{h^2}{2} \frac{\partial f}{\partial x} + \frac{h^2}{2} f \frac{\partial f}{\partial y} + \frac{h^3}{4} f \frac{\partial^2 f}{\partial x \partial y} + \frac{h^3}{8} \frac{\partial^2 f}{\partial x^2}\right)\right. \\
&\quad +2\left(hf + \frac{h^2}{2} \frac{\partial f}{\partial x} + \frac{h^2}{2} f \frac{\partial f}{\partial y} + \frac{h^3}{4}\left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 + f \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\right)\right) \\
&\quad \left. +hf + h^2 \frac{\partial f}{\partial x} + h^2 f \frac{\partial f}{\partial y} + \frac{h^3}{2}\left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 + 2f \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial x^2}\right)\right] \\
&= \frac{1}{6}\left[6hf + 3h^2 \frac{\partial f}{\partial x} + 3h^2 f \frac{\partial f}{\partial y} + 2h^3 f \frac{\partial^2 f}{\partial x \partial y} + h^3 \frac{\partial^2 f}{\partial x^2} + h^3 f \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + h^3 f\left(\frac{\partial f}{\partial y}\right)^2\right] \\
&= hf + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y}\right) + \frac{h^3}{6}\left(2f \frac{\partial^2 f}{\partial x \partial y} + h^3 \frac{\partial^2 f}{\partial x^2} + h^3 f \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + h^3 f\left(\frac{\partial f}{\partial y}\right)^2\right) \cdots
\end{aligned}
$$

This result agrees with the Taylor series (1) through third order. In fact the approximation agrees with the Taylor series through fourth order. See if you can show this. You will need to keep all terms of 4th order in h.