Radiation from a relativistically moving particle

S.M.Lea

Spring 2015

1 Angular distribution of radiation

The Poynting flux is (radgen notes pg 13)

$$\vec{S} = \frac{q^2 \hat{\mathbf{n}}}{4\pi c R^2 \left(1 - \vec{\beta}\cdot\hat{n}\right)^6} \left| \hat{\mathbf{n}} \times \left[\left(\hat{\mathbf{n}} - \vec{\beta} \right) \times \frac{d\vec{\beta}}{dt} \right] \right|^2$$

all evaluated at the retarded time $t_{\text{ret}} = t - R/c$. The energy radiated per unit solid angle in the direction $\hat{\mathbf{n}}$ during a time $\Delta t = T_2 + R(T_2)/c - (T_1 + R(T_1)/c)$ is:

$$\begin{aligned} E &= \int_{T_1 + R(T_1)/c}^{T_2 + R(T_2)/c} R^2 \vec{S}\cdot\hat{n}\, dt \\ &= \int_{T_1}^{T_2} R^2 \vec{S}\cdot\hat{n} \frac{dt}{dt_{ret}} dt_{ret} \end{aligned}$$

The integrand $R^2\vec{S}\cdot\hat{n}\frac{dt}{dt_{ret}}$ is the energy radiated per unit solid angle per unit time along the particle's world line, but measured in the lab frame. We'll call this $dP(t')/d\Omega$.

$$\frac{dt}{dt_{ret}} = \frac{d}{dt_{ret}} \left(t_{ret} + \frac{R(t_{ret})}{c} \right) = 1 + \frac{1}{c}\frac{dR}{dt}$$

Now recall that

$$R = (\vec{x} - \vec{r}) \cdot \hat{n}$$

and so

$$\frac{dR}{dt} = -\vec{v}\cdot\hat{n}$$

and

$$\frac{dt}{dt_{ret}} = 1 - \vec{\beta}\cdot\hat{n}$$

Thus

$$\begin{aligned} \frac{dP(t')}{d\Omega} &= R^2 \vec{S}\cdot\hat{n} \left(1 - \vec{\beta}\cdot\hat{n}\right) \\ &= \frac{q^2}{4\pi c \left(1 - \vec{\beta}\cdot\hat{n}\right)^5} \left| \hat{n} \times \left[\left(\hat{n} - \vec{\beta}\right) \times \frac{d\vec{\beta}}{dt} \right] \right|^2 \end{aligned} \tag{1}$$

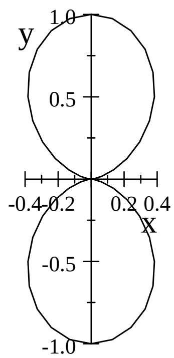
The reduction of the power of $\left(1 - \vec{\beta}\cdot\hat{n}\right)$ in the denominator from 6 to 5 is due to a transformation of angles in going to the lab frame.

The denominator $\left(1-\vec{\beta}\cdot\hat{n}\right)^5$ makes the power radiated very large in the direction of $\vec{\beta}$ when $\beta \approx 1$.

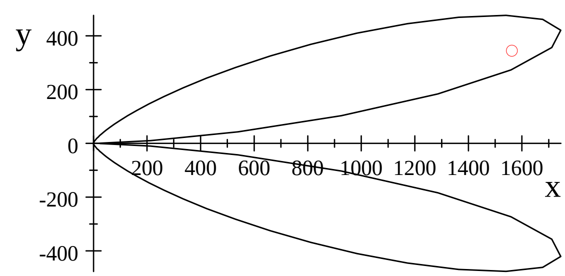
1.1 Acceleration parallel to velocity

Recall that the angular distribution of radiation goes like $\sin^2\theta$ in the non-relativistic case. Thus there is no radiation at all in the direction along the accceleration $\vec{a}$. In the relativistic case, the lobes of radiation are "squashed" in the direction of $\vec{\beta}$. Since $\vec{\beta}\times d\vec{\beta}/dt = 0$, equation (1) simplifies:

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c\left(1-\vec{\beta}\cdot\hat{n}\right)^5}\left|\hat{n}\times\left[\hat{n}\times\frac{d\vec{\beta}}{dt}\right]\right|^2 = \frac{q^2a^2\sin^2\theta}{4\pi c^3\left(1-\beta\cos\theta\right)^5} \tag{2}$$



Non-relativistic



Relativistic: $\beta = 0.9, \gamma = 2.3$

Let's find the angle at which the radiation peaks.

$$\frac{dP(t')}{d\Omega} = k\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

The maximum occurs where the derivative is zero:

$$\frac{2\sin\theta\cos\theta}{\left(1-\beta\cos\theta\right)^5} - 5\frac{\left(\sin^2\theta\right)\left(\beta\sin\theta\right)}{\left(1-\beta\cos\theta\right)^6} = 0$$

which is satisfied for $\theta = 0$ or π (which are the minima where the power radiated is zero, see diagram) or

$$\left(1-\beta\cos\theta\right)2\cos\theta = 5\beta\sin^2\theta = 5\beta\left(1-\cos^2\theta\right)$$

This is a quadratic equation for $\cos\theta$:

$$3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0$$

with solution

$$\cos\theta = \frac{-2\pm\sqrt{4+4\cdot 15\beta^2}}{2\cdot 3\beta} = \frac{-1\pm\sqrt{1+15\beta^2}}{3\beta}$$

Since the square root is > 1, and $|\cos\theta| \le 1$, we must take the plus sign:

$$\cos\theta = \frac{\sqrt{1+15\beta^2}-1}{3\beta}$$

Now if β is very close to 1, we can write $\beta^2 = 1 - 1/\gamma^2$, and expand:

$$\begin{aligned}\cos\theta &= \frac{\sqrt{16-15/\gamma^2}-1}{3\sqrt{1-1/\gamma^2}} = \frac{4}{3}\frac{\left(1-\frac{15}{32\gamma^2}-\frac{1}{4}\right)}{1-\frac{1}{2\gamma^2}} \\ &= \frac{4}{3}\left(\frac{3}{4}-\frac{15}{32\gamma^2}\right)\left(1+\frac{1}{2\gamma^2}\right) \\ &= 1-\frac{1}{8\gamma^2}\end{aligned}$$

Since $\cos\theta \approx 1$, then $\theta \ll 1$, and we may expand the cosine:

$$1-\frac{\theta^2}{2} = 1-\frac{1}{8\gamma^2}$$

and thus

$$\theta = \frac{1}{2\gamma}$$

which is the angle of the maxima of the beam shown in the figure above.

1.2 Velocity perpendicular to acceleration

We choose a coordinate system with $z-$axis along $\vec{\beta}$ and $x-$axis along $\frac{d\vec{\beta}}{dt}$. Then we may write

$$\hat{n} = \cos\theta\hat{\mathbf{z}} + \sin\theta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right)$$
$$\hat{n} - \vec{\beta} = \left(\cos\theta - \beta\right)\hat{\mathbf{z}} + \sin\theta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right)$$

and

$$\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right] = \left(\hat{n}-\vec{\beta}\right)\left(\hat{n}\cdot\frac{d\vec{\beta}}{dt}\right) - \frac{d\vec{\beta}}{dt}\left(1-\vec{\beta}\cdot\hat{n}\right)$$

where

$$\hat{n}\cdot\frac{d\vec{\beta}}{dt} = \dot{\beta}\sin\theta\cos\phi$$

and

$$\vec{\beta}\cdot\hat{n} = \beta\cos\theta$$

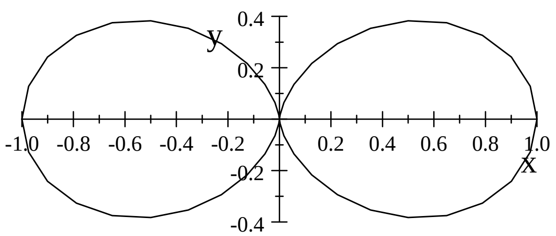
so

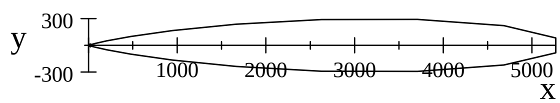
$$
\begin{aligned}
||^2 &= \left|\left[(\cos\theta-\beta)\,\hat{z}+\sin\theta\,(\cos\phi\hat{x}+\sin\phi\hat{y})\right]\dot{\beta}\sin\theta\cos\phi-\dot{\beta}\hat{x}\,(1-\beta\cos\theta)\right|^2 \\
&= \dot{\beta}^2\left[(\cos\theta-\beta)^2\sin^2\theta\cos^2\phi+\sin^4\theta\sin^2\phi\cos^2\phi+\left\{\sin^2\theta\cos^2\phi-(1-\beta\cos\theta)\right\}^2\right] \\
&= \dot{\beta}^2\left[\begin{array}{c}\sin^2\theta\cos^2\phi\cos^2\theta-2\beta\sin^2\theta\cos^2\phi\cos\theta+\beta^2\sin^2\theta\cos^2\phi \\ +\sin^4\theta\sin^2\phi\cos^2\phi+\sin^4\theta\cos^4\phi-2\sin^2\theta\cos^2\phi\,(1-\beta\cos\theta)+(1-\beta\cos\theta)^2\end{array}\right] \\
&= \dot{\beta}^2\left[\begin{array}{c}\sin^2\theta\cos^2\phi\cos^2\theta+\beta^2\sin^2\theta\cos^2\phi \\ +\sin^4\theta\cos^2\phi-2\sin^2\theta\cos^2\phi+(1-\beta\cos\theta)^2\end{array}\right] \\
&= \dot{\beta}^2\left[(1-\beta\cos\theta)^2-\sin^2\theta\cos^2\phi(1-\beta^2)\right]
\end{aligned}
$$

and so

$$
\frac{dP\,(t')}{d\Omega}\propto\frac{1}{(1-\beta\cos\theta)^3}\left(1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2\,(1-\beta\cos\theta)^2}\right)
$$

Distribution in the $\phi=0$ plane:





Non-relativistic

Relativistic: $\gamma=3$

Again we find that the distribution is strongly peaked toward $\theta\sim 0$. Substitute $\beta^2=1-1/\gamma^2$, and assume $\theta\ll 1$. Then

$$
\begin{aligned}
1-\beta\cos\theta &= 1-\sqrt{1-\frac{1}{\gamma^2}}\left(1-\frac{\theta^2}{2}\right) \\
&= 1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right) \\
&= 1-\left(1-\frac{1}{2}\theta^2-\frac{1}{2\gamma^2}\right) \\
&= \frac{1}{2}\theta^2+\frac{1}{2\gamma^2}=\frac{1}{2\gamma^2}\left(1+\gamma^2\theta^2\right)
\end{aligned}
$$

and so

$$
\frac{dP\,(t')}{d\Omega}\propto\frac{\gamma^6}{\left(1+\gamma^2\theta^2\right)^3}\left(1-\frac{4\gamma^2\theta^2\cos^2\phi}{\left(1+\gamma^2\theta^2\right)^2}\right)
$$

The distribution depends on ϕ as well as θ.

$\phi = 0, \pi$

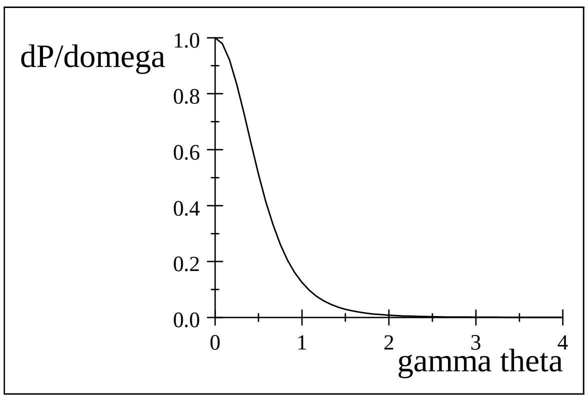
$$\begin{aligned}
\frac{dP(t')}{d\Omega} &\propto \frac{\gamma^6}{\left(1+\gamma^2\theta^2\right)^3}\left(1-\frac{4\gamma^2\theta^2}{\left(1+\gamma^2\theta^2\right)^2}\right) \\
&= \frac{\gamma^6}{\left(1+\gamma^2\theta^2\right)^3}\left(\frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}\right)^2
\end{aligned}$$

which is maximum at $\theta = 0$ and goes to zero at $\theta = 1/\gamma$.

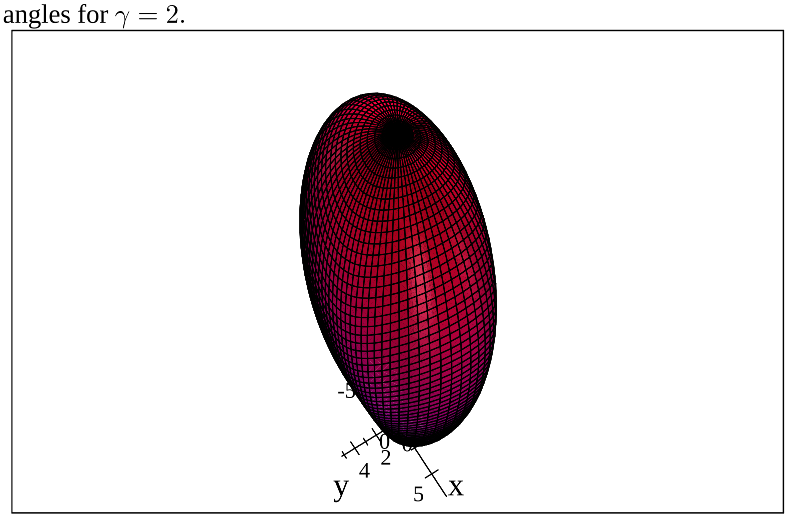
$\phi = \pi/2$

$$\frac{dP(t')}{d\Omega} \propto \frac{\gamma^6}{\left(1+\gamma^2\theta^2\right)^3}$$

which is maximum at $\theta = 0$ and goes to 1/8 of its maximum by $\theta = 1/\gamma$ and to $1/125$ of the maximum at $\theta = 2/\gamma$.



P vs angles for $\gamma = 2$.



In all cases the radiation is beamed along $\vec{\beta}$ to within approximately $1/\gamma$.

2 The power spectrum

The power radiated per unit solid angle is:

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\left|R\vec{E}\right|^2$$

where $\vec{E}$ is evaluated at the retarded time.

The folowing assumptions are justified in any real physical situation:

1. $\vec{E}(t) \to 0$ as $t \to \pm\infty$
2. The charge moves through a small angle while being observed by a fixed observer a large distance away.

Then the total energy radiated per unit solid angle is:

$$\frac{dW}{d\Omega} = \frac{c}{4\pi}\int_{-\infty}^{+\infty}\left|R\vec{E}\right|^2 dt$$

Now we Fourier transform the vector $\vec{F} = R\vec{E}$:

$$\vec{F}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\vec{F}(t)\,e^{i\omega t}dt$$

and by Parseval's theorem, we have:

$$\begin{aligned}\frac{dW}{d\Omega} &= \frac{c}{4\pi}\int_{-\infty}^{+\infty}\left|\vec{F}(\omega)\right|^2 d\omega = \frac{c}{4\pi}\left(\int_{0}^{+\infty}\left|\vec{F}(\omega)\right|^2 d\omega + \int_{-\infty}^{0}\left|\vec{F}(\omega)\right|^2 d\omega\right)\\ &= \frac{c}{4\pi}\int_{0}^{+\infty}\left(\left|\vec{F}(\omega)\right|^2 + \left|\vec{F}(-\omega)\right|^2\right)d\omega\end{aligned}$$

But since $\vec{F}(t)$ is real, then $\vec{F}(-\omega) = \vec{F}^*(\omega)$,and so

$$\frac{dW}{d\Omega} = \frac{c}{2\pi}\int_{0}^{+\infty}\left|\vec{F}(\omega)\right|^2 d\omega$$

and so the energy radiated per unit frequency per unit solid angle is:

$$\frac{d^2W}{d\Omega d\omega} = \frac{c}{2\pi}\left|\vec{F}(\omega)\right|^2$$

Now

$$\vec{F}(t) = \frac{q}{c}\frac{\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right]}{\left(1-\vec{\beta}\cdot\hat{n}\right)^3}$$

and so

$$\vec{F}(\omega) = \frac{q}{c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left.\frac{\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right]}{\left(1-\vec{\beta}\cdot\hat{n}\right)^3}\right|_{t_{\text{ret}}} e^{i\omega t}dt$$

Change variables to $t' = t_{\text{ret}} = t - R(t')/c$. Then $dt' = dt\left(1 - \vec{\beta}\cdot\hat{n}\right)$ and we have:

$$\vec{F}(\omega) = \frac{q}{\sqrt{2\pi}c}\int_{-\infty}^{+\infty} \frac{\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right]}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2}\exp\left[i\omega\left(t' + R(t')/c\right)\right]dt' \tag{3}$$

First let's evaluate $R(t')$. The observation point is at a fixed position $R_0\hat{n}$ from the origin, the radiating charge is at $\vec{r}(t)$, and $\hat{n}\cdot\hat{r} = \cos\chi$. Then

$$R^2 = r^2 + R_0^2 - 2rR_0\cos\chi$$

Since $r \ll R$, we may approximate:

$$R \simeq R_0\sqrt{1 - 2\frac{r}{R_0}\cos\chi} = R_0 - r\cos\chi = R_0 - \hat{n}\cdot\vec{r}$$

Next note that

$$\begin{aligned}
\frac{d}{dt}\frac{\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\vec{\beta}\right)}{1-\vec{\beta}\cdot\hat{n}} &= \frac{\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\frac{d\vec{\beta}}{dt}\right)}{1-\vec{\beta}\cdot\hat{n}} - \left(-\frac{d\vec{\beta}}{dt}\cdot\hat{\mathbf{n}}\right)\frac{\hat{\mathbf{n}}\times\left(\hat{n}\times\vec{\beta}\right)}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2} \\
&= \frac{\hat{\mathbf{n}}\left(\hat{\mathbf{n}}\cdot\frac{d\vec{\beta}}{dt}\right) - \frac{d\vec{\beta}}{dt}}{1-\vec{\beta}\cdot\hat{n}} + \left(\frac{d\vec{\beta}}{dt}\cdot\hat{\mathbf{n}}\right)\frac{\hat{\mathbf{n}}\left(\hat{n}\cdot\vec{\beta}\right) - \vec{\beta}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2} \\
&= \frac{\left(\hat{\mathbf{n}}\cdot\frac{d\vec{\beta}}{dt}\right)\left[\hat{\mathbf{n}}\left(1-\vec{\beta}\cdot\hat{n}\right) + \hat{\mathbf{n}}\left(\hat{n}\cdot\vec{\beta}\right) - \vec{\beta}\right] - \frac{d\vec{\beta}}{dt}\left(1-\vec{\beta}\cdot\hat{n}\right)}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2} \\
&= \frac{\left(\hat{\mathbf{n}}\cdot\frac{d\vec{\beta}}{dt}\right)\left(\hat{\mathbf{n}}-\vec{\beta}\right) - \frac{d\vec{\beta}}{dt}\left(1-\vec{\beta}\cdot\hat{n}\right)}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2} \\
&= \frac{\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right]}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2}
\end{aligned}$$

which is the quantity that appears in the integrand of (3).

Putting it all together, we have:

$$\vec{F}(\omega) = \frac{q}{\sqrt{2\pi}c}e^{i\omega R_0/c}\int_{-\infty}^{+\infty}\frac{d}{dt}\left(\frac{\hat{n}\times\left(\hat{n}\times\vec{\beta}\right)}{1-\vec{\beta}\cdot\hat{n}}\right)\exp\left[i\omega\left(t' - \hat{n}\cdot\vec{r}/c\right)\right]dt'$$

Now we can integrate by parts. The integrated term is:

$$\exp\left[i\omega\left(t' - \hat{n}\cdot\vec{r}/c\right)\right]\frac{\hat{n}\times\left(\hat{n}\times\vec{\beta}\right)}{1-\vec{\beta}\cdot\hat{n}}\Bigg|_{-\infty}^{+\infty}$$

and for any real physical system, we expect $\vec{\beta} \to 0$ as $t \to \pm\infty$, so this term is zero. Then:

$$\vec{F}(\omega) = -\frac{q}{\sqrt{2\pi}c} e^{i\omega R_0/c} \int_{-\infty}^{+\infty} \left(\frac{\hat{n}\times\left(\hat{n}\times\vec{\beta}\right)}{1-\vec{\beta}\cdot\hat{n}} \right) i\omega \left(1-\vec{\beta}\cdot\hat{n}\right) \exp\left[i\omega\left(t'-\hat{n}\cdot\vec{r}/c\right)\right] dt'$$

and finally, renaming $t' = t$, we have:

$$\vec{F}(\omega) = \frac{-i\omega q}{\sqrt{2\pi}c} e^{i\omega R_0/c} \int_{-\infty}^{+\infty} \hat{n}\times\left(\hat{n}\times\vec{\beta}\right) \exp\left[i\omega\left(t-\hat{n}\cdot\vec{r}/c\right)\right] dt$$

and thus

$$\begin{aligned} \frac{d^2W}{d\Omega d\omega} &= \frac{c}{2\pi}\frac{\omega^2 q^2}{2\pi c^2} \left| \int_{-\infty}^{+\infty} \hat{n}\times\left(\hat{n}\times\vec{\beta}\right) \exp\left[i\omega\left(t-\hat{n}\cdot\vec{r}/c\right)\right] dt \right|^2 \\ &= \frac{\omega^2 q^2}{4\pi^2 c} \left| \int_{-\infty}^{+\infty} \hat{n}\times\left(\hat{n}\times\vec{\beta}\right) \exp\left[i\omega\left(t-\hat{n}\cdot\vec{r}/c\right)\right] dt \right|^2 \end{aligned} \qquad (4)$$

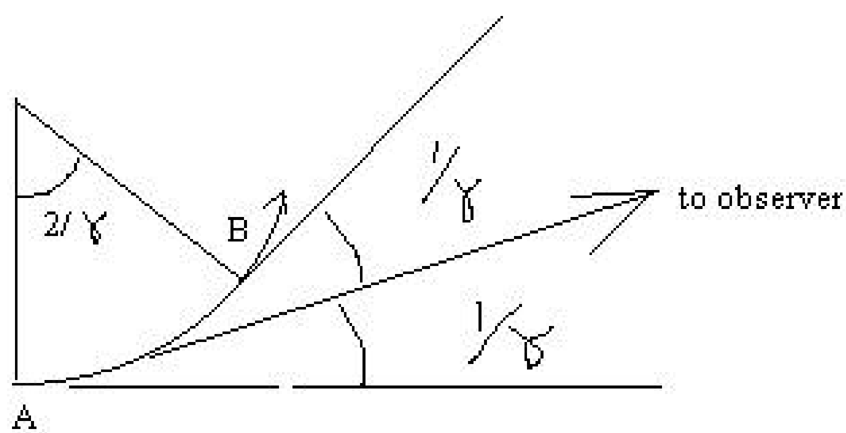
We'll make use of this result in the next section.

3 Synchrotron radiation

3.1 Qualitative discussion

Consider a particle moving along a circular path. The radiation is beamed into a cone of opening angle $\sim 1/\gamma$, as shown above. Thus an observer at P sees radiation while the particle is pointed within an angle $\sim 1/\gamma$ of P, and thus while the particle travels an arc of length $s \sim \frac{2}{\gamma}r$, where r is the radius of the circle. The particle travels this distance in a time

$$\Delta t = \frac{2}{\gamma}\frac{r}{v}$$



Radiation emitted at A at time t_1 reaches $P,$ distance d away, at time $T_1 = \frac{d}{c} + t_1$.

Radiation emitted at B at time $t_2 = t_1 + \Delta t = t_1 + 2r/\gamma v$ reaches P at time $T_2 = \left(d - \frac{2r}{\gamma}\right)/c + t_1 + \frac{2r}{\gamma v}$. The length of the observed pulse is

$$\Delta T = T_2 - T_1 = \frac{2r}{\gamma}\left(\frac{1}{v} - \frac{1}{c}\right)$$

Now $\gamma^2 = \frac{1}{1-\beta^2}$ and so $\beta^2 = 1 - \frac{1}{\gamma^2}$ and then:

$$\frac{1}{\beta} = \left(1 - \frac{1}{\gamma^2}\right)^{-1/2} \simeq 1 + \frac{1}{2\gamma^2}$$

Thus:

$$\Delta T = \frac{2r}{\gamma c}\left(\frac{1}{\beta} - 1\right) = \frac{2r}{\gamma c}\left(\frac{1}{\beta} - 1\right) = \frac{r}{\gamma^3 c}$$

But for an extremely relativistic particle, $\omega_0 = v/r \simeq c/r$, so $\Delta T = 1/\gamma^3\omega_0$. Now a pulse of width ΔT has Fourier components up to at least $1/\Delta T$, and so the observed frequencies extend to at least $\omega_c/3 \equiv \gamma^3\omega_0$.

3.2 Formal treatment

Let the particle's path be in the $x-y-$plane, and let $t = 0$ when the particle is at the origin. Then:

$$\vec{r} = r\left[\sin\omega_0 t\ \hat{\mathbf{x}} + (1 - \cos\omega_0 t)\ \hat{\mathbf{y}}\right]$$
$$\vec{\beta} = \beta\left(\cos\omega_0 t\ \hat{\mathbf{x}} + \sin\omega_0 t\ \hat{\mathbf{y}}\right)$$

Let the observer's direction be in the $x-z-$plane at an angle θ to the $x-$axis. Then:

$$\hat{\mathbf{n}} = \cos\theta\ \hat{\mathbf{x}} + \sin\theta\ \hat{\mathbf{z}}$$

and

$$\begin{aligned}
\hat{n} \times \left(\hat{n} \times \vec{\beta}\right) &= \hat{n}\left(\hat{n} \cdot \vec{\beta}\right) - \vec{\beta} \\
&= \beta\left[\cos\theta\cos\omega_0 t\left(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}\right) - \cos\omega_0 t\hat{\mathbf{x}} - \sin\omega_0 t\hat{\mathbf{y}}\right] \\
&= \beta\left(-\sin^2\theta\cos\omega_0 t\hat{\mathbf{x}} - \sin\omega_0 t\hat{\mathbf{y}} + \sin\theta\cos\theta\cos\omega_0 t\hat{\mathbf{z}}\right)
\end{aligned}$$

and

$$\omega\left(t - \hat{n} \cdot \vec{r}/c\right) = \omega\left(t - \frac{r}{c}\cos\theta\sin\omega_0 t\right)$$

As we saw in the qualitative discussion, the observer will see nothing unless $\theta \lesssim 1/\gamma \ll 1$, and also $\omega_0 t \lesssim 1/\gamma \ll 1$, so we can expand all the sines and cosines to 3rd order in small quantities. (We have to take the quantity that appears in the exponential to higher order than the rest of the integrand.)

$$\begin{aligned}
\omega\left(t - \hat{n} \cdot \vec{r}/c\right) &= \omega\left(t - \frac{r}{c}\left(1 - \frac{\theta^2}{2}\right)\left(\omega_0 t - \frac{\omega_0^3 t^3}{6}\right)\right) \\
&= \omega t\left(1 - \frac{r\omega_0}{c} + \frac{\theta^2\omega_0 r}{2c} + \frac{r\omega_0^3 t^2}{6c}\right)
\end{aligned}$$

Now we assume $\gamma \gg 1$ and thus $\beta = 1 - 1/2\gamma^2$ and note that $\omega_0 r/c = \beta$. Then:

$$\omega\left(t - \hat{n}\cdot\vec{r}/c\right) = \omega\left[t\left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right) + t^3\frac{c^2}{6r^2}\right]$$

which is correct to second order in small quantities.

Similarly

$$\hat{n}\times\left(\hat{n}\times\vec{\beta}\right) = \beta\left(-\omega_0 t\hat{\mathbf{y}} + \theta\hat{\mathbf{z}}\right) = -\omega_0 t\hat{\mathbf{y}} + \theta\hat{\mathbf{z}}$$

to first order in small quantities. So the radiated spectrum is (eqn 4):

$$\begin{aligned}
\frac{d^2W}{d\Omega d\omega} &= \frac{\omega^2 q^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\hat{n}\times\left(\hat{n}\times\vec{\beta}\right)\exp\left[i\omega\left(t-\hat{n}\cdot\vec{r}/c\right)\right]dt\right|^2 \\
&= \frac{\omega^2 q^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\left(-\omega_0 t\hat{\mathbf{y}}+\theta\hat{\mathbf{z}}\right)\exp\left[i\omega\left\{t\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)+t^3\frac{c^2}{6r^2}\right\}\right]dt\right|^2 \\
&= \frac{\omega^2 q^2}{4\pi^2 c}\left|-\omega_0\hat{\mathbf{y}}I_1 + \theta\hat{\mathbf{z}}I_2\right|^2 = \frac{\omega^2 q^2}{4\pi^2 c}\left(\omega_0^2 I_1^2 + \theta^2 I_2^2\right)
\end{aligned}$$

where

$$I_1 = \int_{-\infty}^{+\infty} t\exp\left[\frac{i\omega}{2}\left\{t\left(\frac{1}{\gamma^2}+\theta^2\right)+t^3\frac{c^2}{3r^2}\right\}\right]dt$$

and

$$I_2 = \int_{-\infty}^{+\infty}\exp\left[\frac{i\omega}{2}\left\{t\left(\frac{1}{\gamma^2}+\theta^2\right)+t^3\frac{c^2}{3r^2}\right\}\right]dt$$

To do the integrals, let

$$x = \frac{ct}{r\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}}$$

and

$$\xi = \frac{\omega r}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}$$

Then

$$\frac{3}{2}\xi x = \frac{3}{2}\frac{\omega r}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\frac{ct}{r\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}} = \frac{\omega t}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)$$

and

$$\frac{1}{2}\xi x^3 = \frac{1}{2}\frac{\omega r}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\left(\frac{ct}{r\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}}\right)^3 = \frac{1}{6}\frac{\omega}{r^2}c^2t^3$$

Thus:

$$\begin{aligned}
I_1 &= \frac{r^2}{c^2}\left(\frac{1}{\gamma^2}+\theta^2\right)\int_{-\infty}^{+\infty}x\exp\left[\frac{3}{2}i\xi\left(x+\frac{x^3}{3}\right)\right]dx \\
&= 2i\frac{r^2}{c^2}\left(\frac{1}{\gamma^2}+\theta^2\right)\int_0^{+\infty}x\sin\left[\frac{3}{2}i\xi\left(x+\frac{x^3}{3}\right)\right]dx \\
&= \frac{1}{\omega_0^2}\left(\frac{1}{\gamma^2}+\theta^2\right)\frac{2i}{\sqrt{3}}K_{2/3}\left(\xi\right)
\end{aligned}$$

(cf Jackson 14.78) and similarly

$$\begin{aligned}
I_2 &= \frac{r}{c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\int_{-\infty}^{+\infty}\exp\left[\frac{3}{2}i\xi\left(x+\frac{x^3}{3}\right)\right]dx \\
&= \frac{1}{\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\frac{2}{\sqrt{3}}K_{1/3}(\xi)
\end{aligned}$$

(Jackson 14.78 or G&R 8.433 with $t=\left(\frac{\xi}{2}\right)^{1/3}x$ and G&R's $x=3\left(\frac{\xi}{2}\right)^{2/3}$)

Finally then:

$$\begin{aligned}
\frac{d^2W}{d\Omega d\omega} &= \frac{\omega^2q^2}{4\pi^2c}\left(\omega_0^2I_1^2+\theta^2I_2^2\right) \\
&= \frac{\omega^2q^2}{4\pi^2c}\left(\frac{1}{\gamma^2}+\theta^2\right)^2\frac{4}{3\omega_0^2}\left(\left[K_{2/3}(\xi)\right]^2+\frac{\theta^2}{\frac{1}{\gamma^2}+\theta^2}\left[K_{1/3}(\xi)\right]^2\right) \\
&= \frac{q^2}{3\pi^2c}\frac{\omega^2}{\omega_0^2}\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left(\left[K_{2/3}(\xi)\right]^2+\frac{\theta^2}{\frac{1}{\gamma^2}+\theta^2}\left[K_{1/3}(\xi)\right]^2\right) \qquad (5)
\end{aligned}$$

Now recall that the Bessel functions go like $e^{-\xi}$ for large $\xi=\frac{\omega}{3\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}$, so the emitted power per unit solid angle decreases exponentially for $\omega\gg 3\omega_0\gamma^3\left(1+\gamma^2\theta^2\right)^{-3/2}=\omega_c\left(1+\gamma^2\theta^2\right)^{-3/2}$.

Spectrum at $\theta=0$

For $\omega\ll\omega_c$, we may use the small argument expansion of the Bessel functions.

$$K_\nu(\xi)\sim\frac{\Gamma(\nu)}{2}\left(\frac{2}{\xi}\right)^\nu$$

and note that $\xi=\omega/\omega_c$ when $\theta=0$, so:

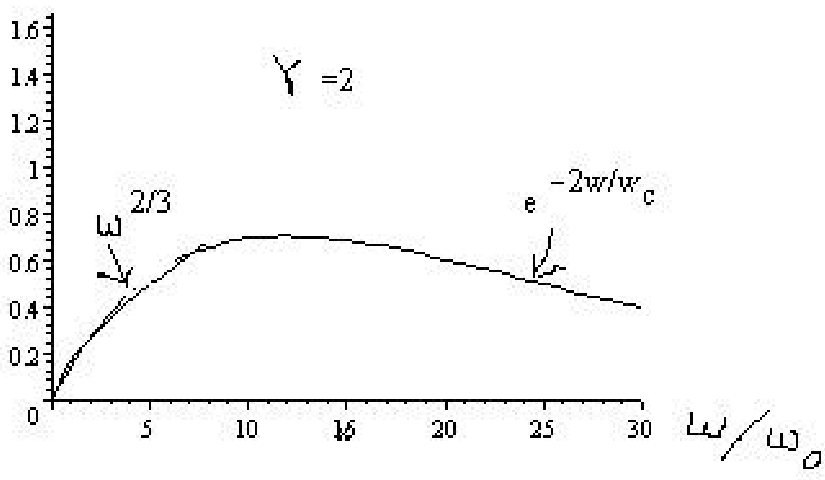
$$\begin{aligned}
\frac{d^2W}{d\Omega d\omega} &= \frac{q^2}{3\pi^2c}\frac{\omega^2}{\omega_0^2}\frac{1}{\gamma^4}\left(\frac{\Gamma(2/3)}{2}\right)^2\left(\frac{2\omega_c}{\omega}\right)^{4/3} \qquad (6) \\
&= \frac{3^{4/3}q^2}{3c}\frac{\omega^2}{\omega_0^2}\frac{2^{4/3-2}\gamma^4}{\gamma^4}\left(\frac{\Gamma(2/3)}{\pi}\right)^2\left(\frac{\omega_0}{\omega}\right)^{4/3} \\
&= \left(\frac{3}{4}\right)^{1/3}\frac{q^2}{c}\left(\frac{\omega}{\omega_0}\right)^{2/3}\left(\frac{\Gamma(2/3)}{\pi}\right)^2
\end{aligned}$$

For $\omega\gg\omega_c$

$$K_\nu(\xi)\sim\sqrt{\frac{\pi}{2\xi}}e^{-\xi}$$

and so:

$$\begin{aligned}
\frac{d^2W}{d\Omega d\omega} &= \frac{q^2}{3\pi^2c}\frac{\omega^2}{\omega_0^2}\frac{1}{\gamma^4}\frac{\pi 3\omega_0\gamma^3}{2\omega}e^{-2\omega/\omega_c} \\
&= \frac{q^2}{\gamma\pi c}\left(\frac{\omega}{\omega_0}\right)e^{-2\omega/\omega_c} \qquad (7)
\end{aligned}$$



Width of the beam

Let's define the angular width of the beam at any frequency by the relation

$$\begin{aligned}\xi(\theta_c) &= \xi(0)+1\\ \frac{\omega}{\omega_c}\left(1+\gamma^2\theta_c^2\right)^{3/2} &= \frac{\omega}{\omega_c}+1\\ \left(1+\gamma^2\theta_c^2\right)^{3/2} &= 1+\frac{\omega_c}{\omega}\end{aligned} \tag{8}$$

At low frequencies, $\omega \ll \omega_c$, the right hand side of (8) is large, and so we must have $\gamma\theta_c \gg 1$. Then

$$(\gamma\theta_c)^3 \simeq \frac{\omega_c}{\omega}$$

or

$$\theta_c \simeq \frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3} \tag{9}$$

which is greater than the "average" value of $1/\gamma$, while for large frequencies, $\omega \gg \omega_c$, the right hand side of (8) is close to 1, and so $\gamma\theta_c \ll 1$. Then:

$$1+\frac{3}{2}\gamma^2\theta_c^2 = 1+\frac{\omega_c}{\omega}$$

or

$$\theta_c = \frac{1}{\gamma}\sqrt{\frac{2\omega_c}{3\omega}} \tag{10}$$

which is less than the average value. In this case we may write

$$e^{-2\xi} = \exp\left(-2\frac{\omega}{\omega_c}\left(1+\gamma^2\theta^2\right)^{3/2}\right) = \exp\left(-\frac{2\omega}{\omega_c}\right)\exp\left(-3\frac{\omega}{\omega_c}\gamma^2\theta^2\right)$$

and thus

$$I(\theta) = I(0)\exp\left(-2\left(\frac{\theta}{\theta_c}\right)^2\right) \tag{11}$$

The beam is a Gaussian.

4 The integrated spectrum

The total energy radiated per unit frequency is:

$$\frac{dW}{d\omega} = \int \frac{d^2W}{d\omega d\Omega} d\Omega = 2\pi \int_{-\pi/2}^{+\pi/2} \frac{d^2W}{d\omega d\Omega} \cos\theta d\theta$$

Notice that our angle variable θ is the lattitude rather than the usual polar angle, hence the $\cos\theta$.

At low frequencies the integrand is slowly varying since the Bessel functions go like small powers of ξ, and thus small powers of the angle θ. Thus, from equations (6) and (9):

$$\begin{aligned}
\frac{dW}{d\omega} &\approx 2\pi\theta_c \frac{d^2W}{d\Omega d\omega}(0) \approx 2\pi\frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3}\left(\frac{3}{4}\right)^{1/3}\frac{q^2}{c}\left(\frac{\omega}{\omega_0}\right)^{2/3}\left(\frac{\Gamma(2/3)}{\pi}\right)^2 \\
&= 2\pi\left(\frac{3\omega_0}{\omega}\right)^{1/3}\left(\frac{3}{4}\right)^{1/3}\frac{q^2}{c}\left(\frac{\omega}{\omega_0}\right)^{2/3}\left(\frac{\Gamma(2/3)}{\pi}\right)^2 \\
&= 2\pi\left(\frac{3}{2}\right)^{2/3}\frac{q^2}{c}\left(\frac{\omega}{\omega_0}\right)^{1/3}\left(\frac{\Gamma(2/3)}{\pi}\right)^2 \\
&= 6\pi\left(\frac{1}{2}\right)^{2/3}\left(\frac{\Gamma(2/3)}{\pi}\right)^2\frac{q^2}{c}\gamma\left(\frac{\omega}{3\gamma^3\omega_0}\right)^{1/3} \\
&= 6\pi\left(\frac{1}{2}\right)^{2/3}\left(\frac{\Gamma(2/3)}{\pi}\right)^2\frac{q^2}{c}\gamma\left(\frac{\omega}{\omega_c}\right)^{1/3}
\end{aligned}$$

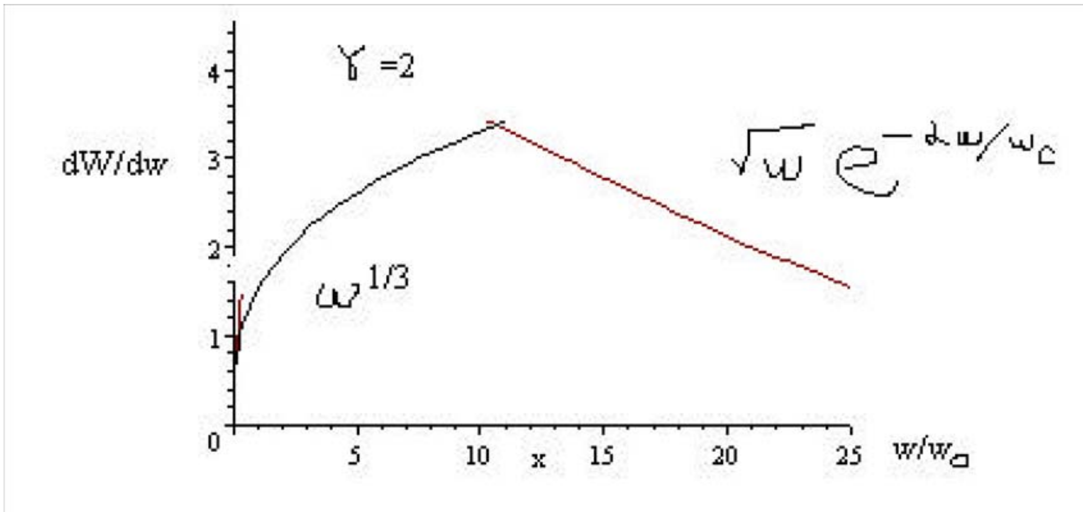
Now the emission is very small for angles $\theta > 1/\gamma$, so we may approximate the integral as:

$$\frac{dW}{d\omega} = 2\pi\int_{-\infty}^{+\infty}\frac{d^2W}{d\omega d\Omega}d\theta$$

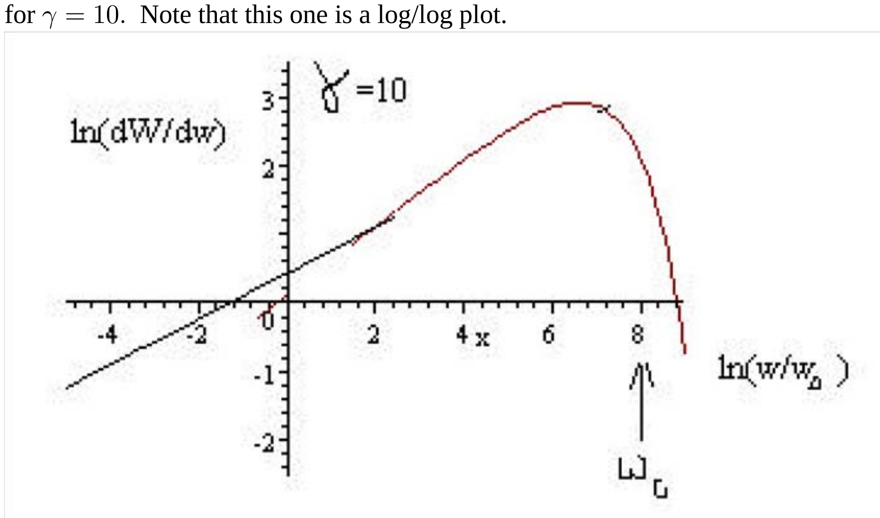
At high frequencies, we use equation (11) and (10):

$$\begin{aligned}
\frac{dW}{d\omega} &= 2\pi\int_{-\infty}^{+\infty}\frac{q^2}{\gamma\pi c}\left(\frac{\omega}{\omega_0}\right)e^{-2\omega/\omega_c}\exp\left(-2\left(\frac{\theta}{\theta_c}\right)^2\right)d\theta \\
&= 2\frac{q^2}{\gamma c}\left(\frac{\omega}{\omega_0}\right)e^{-2\omega/\omega_c}\sqrt{\frac{\pi}{2}}\theta_c \\
&= 2\frac{q^2}{\gamma c}\left(\frac{\omega}{\omega_0}\right)e^{-2\omega/\omega_c}\sqrt{\frac{\pi}{2}}\frac{1}{\gamma}\sqrt{\frac{2\omega_c}{3\omega}} \\
&= \sqrt{3\pi}2\frac{q^2\gamma}{c}\left(\frac{\omega}{3\gamma^3\omega_0}\right)e^{-2\omega/\omega_c}\sqrt{\frac{\omega_c}{3\omega}} \\
&= 2\sqrt{3\pi}\frac{q^2\gamma}{c}\sqrt{\frac{\omega}{\omega_c}}e^{-2\omega/\omega_c}
\end{aligned}$$

Here's what it looks like for $\gamma = 2$



and for $\gamma = 10$. Note that this one is a log/log plot.



Alternate version of page 3

$$\left(\hat{\mathbf{n}}-\tilde{\boldsymbol{\beta}}\right)\times\frac{d\tilde{\boldsymbol{\beta}}}{dt}=\frac{d\beta}{dt}\left[\left(\cos\theta-\beta\right)\hat{\mathbf{y}}-\sin\theta\sin\phi\hat{\mathbf{z}}\right]$$

so

$$\begin{aligned}
\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\tilde{\boldsymbol{\beta}}\right)\times\frac{d\tilde{\boldsymbol{\beta}}}{dt}\right] &= \left[\cos\theta\hat{\mathbf{z}}+\sin\theta\left(\cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}}\right)\right]\times\frac{d\beta}{dt}\left[\left(\cos\theta-\beta\right)\hat{\mathbf{y}}-\sin\theta\sin\phi\hat{\mathbf{z}}\right] \\
&= \frac{d\beta}{dt}\left[\begin{array}{c} -\cos\theta\left(\cos\theta-\beta\right)\hat{\mathbf{x}}+\sin\theta\cos\phi\left(\cos\theta-\beta\right)\hat{\mathbf{z}}+\sin^2\theta\cos\phi\sin\phi\hat{\mathbf{y}} \\ -\sin^2\theta\sin^2\phi\hat{\mathbf{x}} \end{array}\right] \\
&= \frac{d\beta}{dt}\left[\left(-\cos^2\theta-\sin^2\theta\sin^2\phi+\beta\cos\theta\right)\hat{\mathbf{x}}+\sin\theta\cos\phi\left(\cos\theta-\beta\right)\hat{\mathbf{z}}+\sin^2\theta\cos\phi\sin\phi\hat{\mathbf{y}}\right] \\
&= \frac{d\beta}{dt}\left[\left(-1+\sin^2\theta\cos^2\phi+\beta\cos\theta\right)\hat{\mathbf{x}}+\sin\theta\cos\phi\left(\cos\theta-\beta\right)\hat{\mathbf{z}}+\sin^2\theta\cos\phi\sin\phi\hat{\mathbf{y}}\right]
\end{aligned}$$

and so:

$$\begin{aligned}
\left|\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\tilde{\boldsymbol{\beta}}\right)\times\frac{d\tilde{\boldsymbol{\beta}}}{dt}\right]\right|^2 &= \left(\frac{d\beta}{dt}\right)^2\left[\begin{array}{c} \left(-1+\sin^2\theta\cos^2\phi+\beta\cos\theta\right)^2+\sin^2\theta\cos^2\phi\left(\cos\theta-\beta\right)^2 \\ +\sin^4\theta\cos^2\phi\sin^2\phi \end{array}\right] \\
&= \left(\frac{d\beta}{dt}\right)^2\left[\begin{array}{c} \sin^4\theta\cos^4\phi-2\sin^2\theta\cos^2\phi+2\beta\sin^2\theta\cos^2\phi\cos\theta+1-2\beta\cos\theta \\ +\beta^2\cos^2\theta+\sin^2\theta\cos^2\phi\left(\cos^2\theta-2\beta\cos\theta+\beta^2\right) \\ +\sin^4\theta\cos^2\phi\sin^2\phi \end{array}\right] \\
&= \left(\frac{d\beta}{dt}\right)^2\left[\begin{array}{c} \sin^4\theta\cos^2\phi+\sin^2\theta\cos^2\theta\cos^2\phi-2\sin^2\theta\cos^2\phi+1 \\ -2\beta\cos\theta+\beta^2\left(\cos^2\theta+\sin^2\theta\cos^2\phi\right) \end{array}\right] \\
&= \left(\frac{d\beta}{dt}\right)^2\left[1-\sin^2\theta\cos^2\phi-2\beta\cos\theta+\beta^2\cos^2\theta+\beta^2\sin^2\theta\sin^2\phi\right] \\
&= \left(\frac{d\beta}{dt}\right)^2\left[\left(1-\beta\cos\theta\right)^2-\sin^2\theta\cos^2\phi\left(1-\beta^2\right)\right]
\end{aligned}$$