Generation of EM waves

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1 The Green's function

In Lorentz gauge, we obtained the wave equation:

$$\Box^2 A^\beta = \frac{4\pi}{c} J^\beta \tag{1}$$

The corresponding Green's function for the problem satisfies the simpler differential equation

$$\Box^2 D\left(\overset{\leftrightarrow}{x}, \overset{\leftrightarrow}{x}'\right) = \delta^{(4)}\left(\overset{\leftrightarrow}{x} - \overset{\leftrightarrow}{x}'\right)$$

and the boundaries are at infinity. Thus D can depend only on the vector $\overset{\leftrightarrow}{z} = \overset{\leftrightarrow}{x} - \overset{\leftrightarrow}{x}'$. We find D by taking the 4-dimensional Fourier transform:

$$D\left(\overset{\leftrightarrow}{z}\right) = \frac{1}{(2\pi)^2} \int \widetilde{D}(k)\, e^{-i\overset{\leftrightarrow}{k}\cdot\overset{\leftrightarrow}{z}} d^4k$$

(Note that $\overset{\leftrightarrow}{k} \cdot \overset{\leftrightarrow}{z} = \frac{\omega}{c} c\,(t - t') - \vec{k} \cdot (\vec{x} - \vec{x}') = \omega\,(t - t') - \vec{k} \cdot (\vec{x} - \vec{x}')$. This explains the usual convention of transforming the time variable with the opposite sign from that for the space variable.)

Also recall the delta-function integral:

$$\delta^{(4)}\left(\overset{\leftrightarrow}{z}\right) = \frac{1}{(2\pi)^4} \int d^4k e^{-ik\cdot z}$$

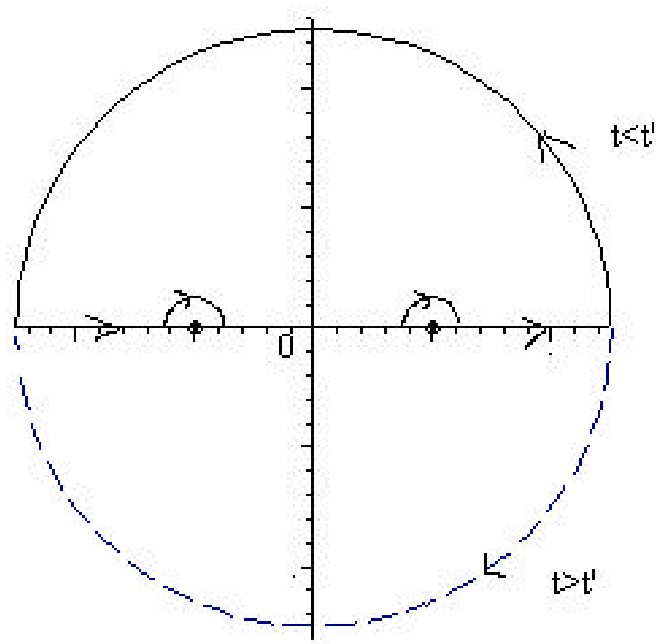
The transformed differential equation reads:

$$-\overset{\leftrightarrow}{k} \cdot \overset{\leftrightarrow}{k}\widetilde{D}\left(\overset{\leftrightarrow}{k}\right) = \frac{1}{(2\pi)^2}$$

and transforming back, we get:

$$
\begin{aligned}
D\left(\overleftrightarrow{z}\right) &= -\frac{1}{(2\pi)^4}\int \frac{e^{-i\overleftrightarrow{k}\cdot\overleftrightarrow{z}}}{\overleftrightarrow{k}\cdot\overleftrightarrow{k}}d^4k \\
&= -\frac{1}{(2\pi)^4}\int d^3k e^{i\vec{k}\cdot(\tilde{\mathbf{x}}-\tilde{\mathbf{x}}')}\int_{-\infty}^{\infty}\frac{d\omega}{c}\frac{e^{-i\omega(t-t')}}{\omega^2/c^2-k^2}
\end{aligned}
$$

We'll do the frequency integral first. For $t<t'$, we close the contour with a big semi-circle in the upper half plane, while for $t>t'$ we close downward. The integrand has two poles, at $\omega=\pm ck$, both on the real axis. Since the result must be zero for $t<t'$ (event precedes its source) we must evelaute the integral along the real axis by deforming the path to go above the two poles.



Contour for inverting the transform

For $t>t'$, we use the lower contour and we traverse it *clockwise*. Each of the poles is simple, and the residues are

$$
\lim_{\omega\to -ck}(\omega+ck)\frac{e^{-i\omega(t-t')}}{\omega^2/c^2-k^2}=c^2\frac{e^{-i(t-t')(-ck)}}{-2ck}=-c\frac{e^{i(t-t')ck}}{2k}
$$

and

$$\lim_{\omega\to ck}\left(\omega-ck\right)\frac{e^{-i\omega(t-t')}}{\omega^2/c^2-k^2}=c^2\frac{e^{-i(t-t')(ck)}}{2ck}=c\frac{e^{-i(t-t')ck}}{2k}$$

Then we have:

$$\begin{aligned}
D\left(\vec{z}\right) &= -\frac{1}{(2\pi)^4}\Theta\left(t-t'\right)\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left(-2\pi i\right)\left(c\frac{e^{-i(t-t')ck}}{2k}-c\frac{e^{i(t-t')ck}}{2k}\right)\\
&= \frac{ic}{2\left(2\pi\right)^3}\Theta\left(t-t'\right)\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{1}{k}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right)
\end{aligned}$$

The first exponential represents an outgoing wave, while the second represents an incoming wave.

Now do the remaining integrals, we choose our polar axis for $\vec{k}$ along the vector $\vec{z}=\vec{x}-\vec{x}'$. Then:

$$\begin{aligned}
D\left(\vec{z}\right) &= \frac{ic}{2\left(2\pi\right)^3}\Theta\left(t-t'\right)\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi}k^2dkd\mu d\phi e^{ikz\mu}\frac{1}{k}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right)\\
&= \frac{ic}{2\left(2\pi\right)^2}\Theta\left(t-t'\right)\int_0^\infty\int_{-1}^{+1}kdkd\mu e^{ikz\mu}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right)\\
&= \frac{ic}{2\left(2\pi\right)^2}\Theta\left(t-t'\right)\int_0^\infty kdk\left.\frac{e^{ikz\mu}}{ikz}\right|_{-1}^{+1}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right)\\
&= \frac{c}{2\left(2\pi\right)^2 z}\Theta\left(t-t'\right)\int_0^\infty dk\left(e^{ikz}-e^{-ikz}\right)\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right)\\
&= \frac{c}{2\left(2\pi\right)^2 z}\Theta\left(t-t'\right)\int_0^\infty dk\left(\begin{array}{c}\exp\left(ik\left(z-c\left(t-t'\right)\right)\right)-\exp ik\left(z+c\left(t-t'\right)\right)\\ -\exp\left[-ik\left(z+c\left(t-t'\right)\right)\right]+\exp -ik\left(z-c\left(t-t'\right)\right)\end{array}\right)\\
&= \frac{c}{2\left(2\pi\right)^2 z}\Theta\left(t-t'\right)\int_{-\infty}^\infty dk\left(-\exp ik\left(z+c\left(t-t'\right)\right)+\exp\left(ik\left(z-c\left(t-t'\right)\right)\right)\right)\\
&= \frac{c}{4\pi z}\Theta\left(t-t'\right)\left(\delta\left(z-c\left(t-t'\right)\right)-\delta\left(z+c\left(t-t'\right)\right)\right)
\end{aligned}$$

The second term can never contribute since both z and $t-t'$ are positive. Thus we finally have:

$$D\left(z\right)=\frac{c}{4\pi z}\Theta\left(t-t'\right)\delta\left(z-c\left(t-t'\right)\right)$$

and thus the solution to equation (1) is:

$$
\begin{aligned}
A^{\alpha}\left(\overleftrightarrow{x}\right) &= \frac{4\pi}{c}\int \frac{c}{4\pi z}\Theta\left(t-t'\right)\delta\left(z-c\left(t-t'\right)\right)J^{\alpha}\left(\overleftrightarrow{x}'\right)d^{4}x' \\
&= \int \frac{1}{z}\Theta\left(t-t'\right)\delta\left(z-c\left(t-t'\right)\right)J^{\alpha}\left(\overleftrightarrow{x}'\right)d^{4}x' \qquad (2)
\end{aligned}
$$

Note: Jackson prefers to write this using the fully covariant expression

$$
\begin{aligned}
\delta\left[\left(\overleftrightarrow{x}-\overleftrightarrow{x}'\right)^{2}\right] &= \delta\left\{\left(ct^{*}\right)^{2}-\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|^{2}\right\} \\
&= \delta\left\{\left(ct^{*}-R\right)\left(ct^{*}+R\right)\right\} \\
&= \frac{1}{2R}\left\{\delta\left(ct^{*}-R\right)+\delta\left(ct^{*}+R\right)\right\}
\end{aligned}
$$

where we wrote $t^{*}=t-t'$, $R=\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|$, and used the result

$$
\delta\left(f\left(x\right)\right)=\sum_{zeros}\frac{\delta\left(x-x_{i}\right)}{\left|f'\left(x_{\prime}\right)\right|}
$$

Thus

$$
D\left(\overleftrightarrow{z}\right)=\frac{1}{2\pi}\Theta\left(t-t'\right)\delta\left[\left(\overleftrightarrow{x}-\overleftrightarrow{x}'\right)^{2}\right]
$$

This is a covariant expression in spite of the explicit appearance of t and t', since the step function serves to locate the result on the forward light cone from the source event, and this is an invariant statement. The expression for the potential becomes:

$$
\begin{aligned}
A^{\alpha}\left(\overleftrightarrow{x}\right) &= \frac{4\pi}{c}\int \frac{1}{2\pi}\Theta\left(t-t'\right)\delta\left[\left(\overleftrightarrow{x}-\overleftrightarrow{x}'\right)^{2}\right]J^{\alpha}\left(\overleftrightarrow{x}'\right)d^{4}x' \\
&= \frac{2}{c}\int \Theta\left(t-t'\right)\delta\left[\left(\overleftrightarrow{x}-\overleftrightarrow{x}'\right)^{2}\right]J^{\alpha}\left(\overleftrightarrow{x}'\right)d^{4}x' \qquad (3)
\end{aligned}
$$

2 Charge density and current for a point charge.

The charge and current densities are given by

$$
\rho\left(\vec{x},t\right)=q\delta\left(\vec{x}-\vec{r}\left(t\right)\right)
$$

and

$$\vec{J}\left(\vec{x},t\right)=q\vec{v}\delta\left(\vec{x}-\vec{r}\left(t\right)\right)$$

and so the 4-vector is

$$J^{\alpha}=\left(c\rho,\vec{v}\right)=q\left(c,\vec{v}\right)\delta\left(\vec{x}-\vec{r}\left(t\right)\right)=\frac{q}{\gamma}u^{\alpha}\delta\left(\vec{x}-\vec{r}\left(t\right)\right) \tag{4}$$

We can write this covariantly as

$$J^{\alpha}\left(t_0,\vec{x}\right)=qc\int u^{\alpha}\delta\left(\overleftrightarrow{x}-\overleftrightarrow{x}_0\left(\tau\right)\right)d\tau \tag{5}$$

where

$$x_0^{\alpha}=\left(ct_0,\vec{r}\left(t_0\right)\right)$$

is the charge's position vector at time t_0. To see why this works, note that

$$\delta\left(\overleftrightarrow{x}-\overleftrightarrow{x}_0\left(\tau\right)\right)=\delta\left(ct-ct_0\left(\tau\right)\right)\delta\left(\vec{x}-\vec{r}\left(\tau\right)\right)$$

and

$$\delta\left(ct-ct_0\left(\tau\right)\right)=\frac{\delta\left(\tau\left(t\right)-\tau\left(t_0\right)\right)}{\left|cdt/d\tau\right|}=\frac{\delta\left(\tau\left(t\right)-\tau\left(t_0\right)\right)}{c\gamma}$$

Thus

$$\begin{aligned}J^{\alpha}\left(t_0,\vec{x}\right) &= qc\int u^{\alpha}\frac{\delta\left(\tau\left(t\right)-\tau\left(t_0\right)\right)}{c\gamma}\delta\left(\vec{x}-\vec{r}\left(\tau\right)\right)d\tau\\ &= \frac{q}{\gamma}u^{\alpha}\delta\left(\vec{x}-\vec{r}\left(t_0\right)\right)\end{aligned}$$

which agrees with equation (4).

3 Radiation from a moving charge- the potential

We insert the current vector (5) into the expression (3) for the potential:

$$\begin{aligned}A^{\alpha}\left(\overleftrightarrow{x}\right) &= \frac{2}{c}\int\Theta\left(t-t'\right)\delta\left[\left(\overleftrightarrow{x}-\overleftrightarrow{x}'\right)^2\right]qc\int u^{\alpha}\delta\left(\overleftrightarrow{x}'-\overleftrightarrow{x}_0\left(\tau\right)\right)d\tau d^4x'\\ &= 2q\int\Theta\left(t-t'\right)\delta\left[\left(\overleftrightarrow{x}-\overleftrightarrow{x}'\right)^2\right]\int u^{\alpha}\delta\left(\overleftrightarrow{x}'-\overleftrightarrow{x}_0\left(\tau\right)\right)d\tau d^4x'\end{aligned}$$

Using the delta function, the integration over $\overleftrightarrow{x}'$ is easy:

$$A^{\alpha}\left(\overleftrightarrow{x}\right) = 2q\int \Theta\left(t - t_0\left(\tau\right)\right)\delta\left[\left(\overleftrightarrow{x} - \overleftrightarrow{x}_0\left(\tau\right)\right)^2\right]u^{\alpha}d\tau$$

Now again we need to evaluate the delta function of a function- here the function $f\left(\tau\right) = \left(\overleftrightarrow{x} - \overleftrightarrow{x}_0\left(\tau\right)\right)^2 = \left(x^{\alpha} - x_0^{\alpha}\right)\left(x_{\alpha} - x_{0\alpha}\right)$. The derivative

$$f'\left(\tau\right) = -2\frac{dx_0^{\alpha}}{d\tau}\left(x_{\alpha} - x_{0\alpha}\right) = -2v^{\alpha}\left(x_{\alpha} - x_{0\alpha}\right)$$

and the zeros are those values of τ for which $\left(x^{\alpha} - x_0^{\alpha}\right)\left(x_{\alpha} - x_{0\alpha}\right) = 0$. Expanding this out, we get

$$\begin{aligned} c\left(t - t_0\right)^2 - \left(\vec{x} - \vec{r}\left(t_0\right)\right)^2 &= 0 \\ t - t_0 &= R/c \end{aligned}$$

where

$$R = \left|\vec{x} - \vec{r}\left(t_0\right)\right|$$

Only the positive value contributes to the integral because of the factor $\Theta\left(t - t_0\left(\tau\right)\right)$. (The event is on the positive light cone from the source.) Thus:

$$t = t_0 + R/c$$

or

$$t_0 = t - R/c = t_{\text{ret}} \tag{6}$$

the *retarded time.*

Then the denominator is

$$2v^{\alpha}\left(x_{\alpha} - x_{0\alpha}\right) = 2\left(\gamma c, \gamma\vec{v}\right)\cdot\left(c\left(t - t_0\right), -\vec{R}\right) = 2\left(\gamma c^2\left(t - t_0\right) - \gamma\vec{v}\cdot\vec{R}\right)$$

or, using equation (6),

$$2\left(\gamma c^2 R/c - \gamma\vec{v}\cdot\vec{R}\right) = 2\gamma Rc\left(1 - \vec{\beta}\cdot\hat{R}\right)$$

Then

$$
\begin{aligned}
A^{\alpha}\left(\overleftrightarrow{x}\right) &= 2q\int \Theta\left(t-t_0\left(\tau\right)\right)\delta\left[\left(\overleftrightarrow{x}-\overleftrightarrow{x}_0\left(\tau\right)\right)^2\right]u^{\alpha}\left(\tau\right)d\tau \\
&= \left.\frac{q\gamma\left(c,\vec{v}\right)}{\gamma Rc\left(1-\vec{\beta}\cdot\hat{R}\right)}\right|_{t_{\mathrm{ret}}} \\
&= \left.\frac{q\left(1,\vec{\beta}\right)}{R\left(1-\vec{\beta}\cdot\hat{R}\right)}\right|_{t_{\mathrm{ret}}}
\end{aligned}
$$

These are the Lienard-Wiechert potentials. They may also be written covariantly as:

$$
\begin{aligned}
A^{\alpha} &= q\frac{u^{\alpha}}{u^{\beta}\triangle x_{\beta}} \qquad (7)\\
&= q\frac{\gamma\left(c,\vec{v}\right)}{\gamma\left(c,\vec{v}\right)\cdot\left(c\left(t-t_0\right),\vec{x}-\vec{x}_0\right)} \\
&= q\frac{\left(1,\vec{\beta}\right)}{c\left(t-t_0\right)-\vec{\beta}\cdot\vec{R}} = \frac{q}{R}\frac{\left(1,\vec{\beta}\right)}{\left(1-\vec{\beta}\cdot\hat{R}\right)}
\end{aligned}
$$

using (6), and with

$$\triangle x_{\beta} = x_{\beta} - x_{0\beta}$$

all evaluated at the retarded time.

4 Fields due to a moving charge

To compute the fields from the potentials (2 or 7), we need to take derivatives. That is, we need to compare the potentials at two nearby events. The potential A^{α} changes as x^{α} changes both because of its explicit dependence on x^{α} but also because of the implicit dependence on $\tau_0\left(x^{\alpha}\right)$ – that is because of the need to look back along the light cone to find the source event.

$$dA^{\mu} = \frac{\partial A^{\mu}}{\partial x^{\nu}}dx^{\nu} + \frac{dA^{\mu}}{d\tau_0}d\tau_0 \qquad (8)$$

Differentiating the light cone constraint $(x^{\alpha}-r^{\alpha}(\tau_0))(x_{\alpha}-r_{\alpha}(\tau_0))=0$, we get:

$$2(x^{\alpha}-r^{\alpha}(\tau_0))(dx_{\alpha}-dr_{\alpha}(\tau_0))=0 \tag{9}$$

and from the definition of the particle's velocity

$$dr_{\alpha}(\tau_0)=u_{\alpha}d\tau_0$$

So

$$\begin{aligned}(x^{\alpha}-r^{\alpha}(\tau_0))(dx_{\alpha}-u_{\alpha}d\tau_0) &= 0\\ d\tau_0 &= \frac{(x^{\alpha}-r^{\alpha}(\tau_0))\,dx_{\alpha}}{(x^{\beta}-r^{\beta}(\tau_0))\,u_{\beta}}\\ &= \frac{\Delta x^{\alpha}dx_{\alpha}}{\Delta x^{\beta}u_{\beta}}\end{aligned} \tag{10}$$

where I defined $\Delta x^{\alpha}\equiv(x^{\alpha}-r^{\alpha}(\tau_0))$

Thus from equation (8 and 7), we find

$$\begin{aligned}dA^{\mu} &= \partial_{\nu}\left(\frac{qu^{\mu}}{u^{\beta}\Delta x_{\beta}}\right)dx^{\nu}+\frac{d}{d\tau_0}\left(\frac{qu^{\mu}}{u^{\beta}\Delta x_{\beta}}\right)d\tau_0\\ &= qu^{\mu}\left\{\frac{-u^{\beta}\partial_{\nu}\Delta x_{\beta}}{[u^{\gamma}\Delta x_{\gamma}]^2}\right\}dx^{\nu}+q\left\{\frac{a^{\mu}}{u^{\beta}\Delta x_{\beta}}-u^{\mu}\frac{a^{\beta}\Delta x_{\beta}+u^{\beta}(-u_{\beta})}{[u^{\gamma}\Delta x_{\gamma}]^2}\right\}d\tau_0\end{aligned}$$

Note that

$$\partial_{\nu}\Delta x_{\beta}=\frac{\partial}{\partial x^{\nu}}(x_{\beta}-r_{\beta}(\tau_0))=\frac{\partial}{\partial x^{\nu}}g_{\beta\alpha}(x^{\alpha}-r^{\alpha}(\tau_0))=g_{\beta\alpha}\delta^{\alpha}_{\nu}=g_{\beta\nu}$$

Now we insert the expression (10) for $d\tau_0$, and factor a bit:

$$\begin{aligned}dA^{\mu} &= \frac{qdx^{\nu}}{[u^{\gamma}\Delta x_{\gamma}]^2}\left\{-u^{\mu}u^{\beta}g_{\beta\nu}+\Delta x_{\nu}\left(a^{\mu}-u^{\mu}\frac{a^{\beta}\Delta x_{\beta}-u^{\beta}u_{\beta}}{u^{\varepsilon}\Delta x_{\varepsilon}}\right)\right\}\\ &= \frac{qdx^{\nu}}{[u^{\gamma}\Delta x_{\gamma}]^2}\left\{-u^{\mu}u_{\nu}+\Delta x_{\nu}\left(a^{\mu}+u^{\mu}\frac{c^2-a^{\beta}\Delta x_{\beta}}{u^{\varepsilon}\Delta x_{\varepsilon}}\right)\right\}\end{aligned}$$

So

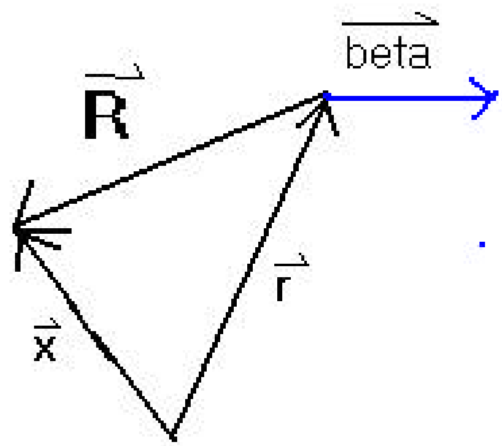
$$\frac{\partial A^{\mu}}{\partial x^{\nu}}=\frac{q}{[u^{\gamma}\Delta x_{\gamma}]^2}\left\{-u^{\mu}u_{\nu}+\Delta x_{\nu}\left(a^{\mu}+u^{\mu}\frac{c^2-a^{\beta}\Delta x_{\beta}}{u^{\varepsilon}\Delta x_{\varepsilon}}\right)\right\}$$

Now we compute the field tensor components

$$F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}$$

The symmetric term $u^{\mu}u^{\nu}$ will cancel in the subtraction, leaving

$$F^{\mu\nu} = \frac{q}{\left[u^{\gamma}\Delta x_{\gamma}\right]^2}\left\{\Delta x^{\mu}a^{\nu} - \Delta x^{\nu}a^{\mu} + \frac{c^2 - a^{\beta}\Delta x_{\beta}}{u^{\varepsilon}\Delta x_{\varepsilon}}\left(u^{\nu}\Delta x^{\mu} - u^{\mu}\Delta x^{\nu}\right)\right\} \quad (11)$$



Now, using the notation in the diagram above,

$$\begin{aligned} \overset{\leftrightarrow}{\Delta x} &= R\left(1, \tilde{\mathbf{n}}\right) \\ \overset{\leftrightarrow}{u} &= c\gamma\left(1, \tilde{\boldsymbol{\beta}}\right) \end{aligned}$$

and

$$\overset{\leftrightarrow}{a} = \frac{d\overset{\leftrightarrow}{u}}{d\tau} = \gamma\frac{d\overset{\leftrightarrow}{u}}{dt} = c\gamma\left(\frac{d\gamma}{dt}, \tilde{\boldsymbol{\beta}}\frac{d\gamma}{dt} + \gamma\frac{d\tilde{\boldsymbol{\beta}}}{dt}\right)$$

and

$$\frac{d\gamma}{dt} = \gamma^3\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}$$

so

$$\overleftrightarrow{a} = c\gamma^2 \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt}, \gamma^2 \left(\vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \vec{\beta} + \frac{d\vec{\beta}}{dt} \right) \tag{12}$$

Then we can compute the dot products

$$u^\gamma \Delta x_\gamma = c\gamma R \left(1 - \vec{\beta} \cdot \hat{n} \right)$$

and

$$\begin{aligned}
a^\beta \Delta x_\beta &= c\gamma^2 \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt}, \gamma^2 \left(\vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \vec{\beta} + \frac{d\vec{\beta}}{dt} \right) \cdot R\left(1, \vec{n}\right) \\
&= c\gamma^2 R \left(\gamma^2 \vec{\beta} \cdot \frac{d\boldsymbol{\vec{\beta}}}{dt} - \gamma^2 \left(\vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \vec{\beta} \cdot \hat{n} - \frac{d\vec{\beta}}{dt} \cdot \hat{n} \right) \\
&= c\gamma^2 R \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \left(1 - \vec{\beta} \cdot \hat{n} \right) - \frac{d\vec{\beta}}{dt} \cdot \hat{n} \right)
\end{aligned}$$

Then from (11), and writing

$$K = \frac{q}{\left[c\gamma R \left(1 - \vec{\beta} \cdot \hat{n} \right) \right]^2} \tag{13}$$

we have

$$\begin{aligned}
F^{\mu\nu} &= K \left\{ \begin{array}{c} \Delta x^\mu a^\nu - \Delta x^\nu a^\mu + \\ \frac{\left(u^\nu \Delta x^\mu - u^\mu \Delta x^\nu\right)}{c\gamma R\left(1 - \vec{\beta} \cdot \hat{n}\right)} \left(c^2 - c\gamma^2 R \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \left(1 - \vec{\beta} \cdot \hat{n} \right) - \frac{d\vec{\beta}}{dt} \cdot \hat{n} \right) \right) \end{array} \right\} \\
&= K \left\{ \Delta x^\mu a^\nu - \Delta x^\nu a^\mu + \left(u^\nu \Delta x^\mu - u^\mu \Delta x^\nu\right) \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R \left(1 - \vec{\beta} \cdot \hat{n} \right)} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \right\}
\end{aligned} \tag{14}$$

The top row is

$$
\begin{aligned}
F^{0i} &= K\left\{\Delta x^0 a^i - \Delta x^i a^0 + \left(u^i \Delta x^0 - u^0 \Delta x^i\right)\left(\frac{c+\gamma^2 R \frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \gamma^3 \vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\right\} \\
&= K\left\{R\left(a^i - n^i c\gamma^4 \vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right) + c\gamma R\left(\beta^i - n^i\right)\left(\frac{c+\gamma^2 R \frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \gamma^3 \vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\right\} \\
&= KcR\left\{\gamma^2\left(\gamma^2\left(\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\beta^i + \frac{d\beta^i}{dt}\right) - n^i\gamma^4\vec{\beta}\cdot\frac{d\vec{\beta}}{dt} + \left(\beta^i - n^i\right)\left(\frac{c+\gamma^2 R \frac{d\vec{\beta}}{dt}\cdot\hat{n}}{R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \gamma^4 \vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\right\}
\end{aligned}
$$

where we used (12) for $\vec{a}$. Then, inserting (13), the electric field is given by

$$
\begin{aligned}
E^i &= -F^{0i} \\
&= \gamma^2 KcR\left\{\left(n^i - \beta^i\right)\left(\gamma^2\vec{\beta}\cdot\frac{d\vec{\beta}}{dt} + \frac{c+\gamma^2 R \frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma^2 R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \gamma^2 \vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right) - \frac{d\beta^i}{dt}\right\} \\
&= \frac{q}{cR\left(1-\vec{\beta}\cdot\hat{n}\right)^2}\left\{\left(n^i - \beta^i\right)\frac{c+\gamma^2 R \frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma^2 R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \frac{d\beta^i}{dt}\right\} \qquad (15)
\end{aligned}
$$

We may divide this result into two parts:

$$
E^i_{\text{coulomb}} = \frac{q}{\gamma^2 R^2\left(1-\vec{\beta}\cdot\hat{n}\right)^3}\left(n^i - \beta^i\right)
$$

This field has the usual $1/R^2$ dependence and is in fact the same as Jackson's equation 11.154. (See Jackson page 664 for the details.) The other term is the radiation field: it depends on the particle's acceleration.

$$
E^i_{\text{rad}} = \frac{q}{cR\left(1-\vec{\beta}\cdot\hat{n}\right)^2}\left\{\left(n^i - \beta^i\right)\frac{\frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\left(1-\vec{\beta}\cdot\hat{n}\right)} - \frac{d\beta^i}{dt}\right\} \qquad (16)
$$

It decreases as $1/R$ rather than $1/R^2$, and so dominates the Coulomb field at large distances. We can simplify the expression by noting that:

$$
\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right] = \left(\hat{n}-\vec{\beta}\right)\left(\hat{n}\cdot\frac{d\vec{\beta}}{dt}\right) - \frac{d\vec{\beta}}{dt}\left(1-\vec{\beta}\cdot\hat{n}\right)
$$

Thus

$$\vec{E}_{\text{rad}} = \frac{q}{cR\left(1-\vec{\beta}\cdot\hat{n}\right)^3}\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right] \tag{17}$$

Remember: everything is evaluated at $t_{\text{retarded.}}$ Also notice that $\vec{E}_{\text{rad}}$ is perpendicular to $\hat{n}$, and is proportional to the acceleration $d\vec{\beta}/dt$.

The magnetic field is given by:

$$\vec{B} = \left(-F^{23}, F^{13}, -F^{12}\right)$$

For example:

$$\begin{aligned}
B_x &= -F^{23} \\
&= -K\left\{\Delta x^2 a^3 - \Delta x^3 a^2 + \left(u^3\Delta x^2 - u^2\Delta x^3\right)\left(\frac{c+\gamma^2 R\frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \gamma^3\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\right\} \\
&= -K\left\{n^2 a^3 - n^3 a^2 + \left(u^3 n^2 - u^2 n^3\right)\left(\frac{c+\gamma^2 R\frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \gamma^3\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\right\} \\
&= \frac{-q}{c\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)^2}\left\{\begin{array}{c}\gamma^3\left(\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\left(n^2\beta^3 - n^3\beta^2\right) + \gamma\left(n^2\frac{d\beta^3}{dt} - n^3\frac{d\beta^2}{dt}\right) \\ +\left(\beta^3 n^2 - \beta^2 n^3\right)\left(\frac{c+\gamma^2 R\frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)} - \gamma^3\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)\end{array}\right\}
\end{aligned}$$

The first and last terms cancel. Thus:

$$\vec{B} = \frac{-q}{c\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)^2}\left\{\frac{c+\gamma^2 R\frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\gamma R\left(1-\vec{\beta}\cdot\hat{n}\right)}\hat{n}\times\vec{\beta} + \gamma\hat{n}\times\frac{d\vec{\beta}}{dt}\right\}$$

Again we may divide the result into two parts:

$$\vec{B}_{\text{B-S}} = \frac{q}{\gamma^2 R^2\left(1-\vec{\beta}\cdot\hat{n}\right)^3}\vec{\beta}\times\hat{n}$$

This term, which goes as $1/R^2$, is the usual Biot -Savart law result, with relativistic corrections. (Compare J eqn 11.152 and our expression for $\vec{E}_{\text{coulomb}}$)

The radiation term is:

$$
\begin{aligned}
\vec{B}_{\mathrm{rad}} &= \frac{-q}{cR\left(1-\vec{\beta}\cdot\hat{n}\right)^2}\left\{\frac{\frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\left(1-\vec{\beta}\cdot\hat{n}\right)}\hat{n}\times\vec{\beta}+\hat{n}\times\frac{d\vec{\beta}}{dt}\right\} \\
&= -\hat{n}\times\frac{q}{cR\left(1-\vec{\beta}\cdot\hat{n}\right)^2}\left\{\frac{\frac{d\vec{\beta}}{dt}\cdot\hat{n}}{\left(1-\vec{\beta}\cdot\hat{n}\right)}\vec{\beta}+\frac{d\vec{\beta}}{dt}\right\} \\
&= \hat{n}\times\vec{E}_{\mathrm{rad}}
\end{aligned}
$$

where we used equation (16) for the electric field.

5 Radiated power

The Poynting vector for the radiation field is

$$
\begin{aligned}
\vec{S} &= \frac{c}{4\pi}\vec{E}_{\mathrm{rad}}\times\vec{B}_{\mathrm{rad}} \\
&= \frac{c}{4\pi}\vec{E}_{\mathrm{rad}}\times\left(\hat{n}\times\vec{E}_{\mathrm{rad}}\right) \\
&= \frac{c}{4\pi}E_{\mathrm{rad}}^2\hat{n} \\
&= \frac{c}{4\pi}\left(\frac{q}{cR\left(1-\vec{\beta}\cdot\hat{n}\right)^3}\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right]\right)^2\hat{n} \\
&= \frac{q^2}{4\pi R^2 c\left(1-\vec{\beta}\cdot\hat{n}\right)^6}\left|\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right]\right|^2\hat{n}
\end{aligned}
$$

Thus the power radiated per unit solid angle is:

$$
\begin{aligned}
\frac{dP}{d\Omega} &= R^2\vec{S}\cdot\hat{n} \\
&= \frac{q^2}{4\pi c\left(1-\vec{\beta}\cdot\hat{n}\right)^6}\left|\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\frac{d\vec{\beta}}{dt}\right]\right|^2 \qquad (18)
\end{aligned}
$$

For non-relativistic motion, $\beta \ll 1$, we retrieve the Larmor formula (704 wavemks notes eqn 35 with a units change):

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{q^2}{4\pi c}\left|\hat{n}\times\left[\hat{n}\times\frac{d\vec{\beta}}{dt}\right]\right|^2 \\
&= \frac{q^2}{4\pi c^3}a^2\sin^2\theta
\end{aligned}$$

where $a = dv/dt$ is the usual 3-acceleration and θ is the angle between $\vec{a}$ and $\hat{n}$. The total power radiated in the non-relativistic case is:

$$\begin{aligned}
P &= \int \frac{dP}{d\Omega}d\Omega = \int_{-1}^{+1} d\mu \int_0^{2\pi} d\phi \frac{q^2}{4\pi c^3}a^2\left(1-\mu^2\right) \\
&= \frac{q^2}{2c^3}a^2\left(\mu - \frac{\mu^3}{3}\right)\Bigg|_{-1}^{+1} = \frac{2q^2}{3c^3}a^2 \qquad (19)
\end{aligned}$$

When β is not small, there are important changes. The denominator of equation (18) gets small when $\vec{\beta}\cdot\hat{n}$ is close to 1, that is for direction of propagation close to the particle's velocity $\vec{\beta}$. Thus the radiation is strongly beamed along the direction of the particle's motion.

5.1 Covariant generalization

From problem 12.15 we know that $\int \Theta^{0\alpha}dV$ transforms as a vector. Thus $\int \Theta^{00}dV$ transforms in the same way as $x^0 = ct$. Thus the ratio, which is the power, is a Lorentz invariant. But in the instantaneous rest frame of the particle,

$$a^\alpha a_\alpha = -\vec{a}\cdot\vec{a} = -a^2$$

So we can write eqn (19) as

$$\begin{aligned}
P &= -\frac{2}{3}\frac{q^2}{c^3}a^\alpha a_\alpha \\
&= -\frac{2}{3}\frac{q^2}{c}\left(\dot{\gamma}^2 - \left(\dot{\gamma}\vec{\beta} + \gamma\frac{d\vec{\beta}}{d\tau}\right)^2\right)
\end{aligned}$$

where dot means $d/d\tau$. The term is parentheses simplifies as follows:

$$
\begin{aligned}
\dot{\gamma}^2 - \left(\dot{\gamma}\vec{\beta} + \gamma\frac{d\vec{\beta}}{d\tau}\right)^2 &= \dot{\gamma}^2 - \left(\dot{\gamma}^2\beta^2 + \gamma^2\dot{\vec{\beta}}^2 + 2\gamma\dot{\gamma}\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}\right) \\
&= \frac{1}{\gamma^2}\dot{\gamma}^2 - \gamma^2\dot{\vec{\beta}}^2 - 2\gamma\dot{\gamma}\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}
\end{aligned}
$$

But

$$\dot{\gamma} = \gamma^3\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}$$

So

$$
\begin{aligned}
() &= \gamma^4\left(\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}\right)^2 - \gamma^2\dot{\vec{\beta}}^2 - 2\gamma^4\left(\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}\right)^2 \\
&= -\gamma^2\left(\dot{\vec{\beta}}^2 + \gamma^2\left(\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}\right)^2\right)
\end{aligned}
$$

Giving

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^2\left(\dot{\vec{\beta}}^2 + \gamma^2\left(\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}\right)^2\right)$$

which is positive, as expected. Now let's look at the two special cases:

1. $\vec{\beta}$ parallel to $d\vec{\beta}/dt$:

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^2\dot{\beta}^2\left(1 + \frac{\beta^2}{1-\beta^2}\right) = \frac{2}{3}\frac{q^2}{c}\gamma^4\dot{\beta}^2$$

where $\dot{\beta} = d\beta/d\tau = \gamma d\beta/dt = \gamma a/c$. Thus

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^6 a^2 \tag{20}$$

2. $\vec{\beta}$ perpendicular to $d\vec{\beta}/dt$:

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^2\dot{\vec{\beta}}^2 = \frac{2}{3}\frac{q^2}{c^3}\gamma^4 a^2 \tag{21}$$

Thus we get more radiated power per unit acceleration if $\vec{a}$ is parallel to $\vec{\beta}$ (linear motion). However, if we relate the power to the force acting on the particle, (see exam problem),we get a different interpretation.

1. $\vec{F}$ parallel to $\vec{\beta}$. In this case $a = F/m\gamma^3$ and so

$$P = \frac{2}{3}\frac{q^2}{c^3}\left(\frac{F}{m}\right)^2$$

2. $\vec{F}$ perpendicular to $\vec{\beta}$. In this case $a = F/m\gamma$ and so

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^2\left(\frac{F}{m}\right)^2$$

So the power radiated per unit force is greater when $\vec{F}$ is perpendicular to $\vec{\beta}$ (circular motion).