Wave propagation

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1 A wave signal

An electromagnetic wave signal may be represented by a function of space and time $u(\vec{x},t)$ where u might be one component of the electric field, for example. Then we can write this function in terms of its Fourier transform:

$$u(\vec{x},t)=\frac{1}{(2\pi)^2}\int_{\text{all }\omega\text{ and }k\text{ space}} A(\vec{k},\omega)\exp\left(i\vec{k}\cdot\vec{x}-i\omega t\right)d\omega d\vec{k}$$

The dispersion relation for the wave gives a relation (such as eqn 12 in the plasma wave notes) between ω and $\vec{k}$ which allows us to write the integral in terms of $\vec{k}$ alone[1]:

$$u(\vec{x},t)=\frac{1}{(2\pi)^2}\int_{\text{all }k\text{ space}} A(\vec{k})\exp\left[i\vec{k}\cdot\vec{x}-i\omega\left(\vec{k}\right)t\right]\,d\vec{k}$$

Now let's simplify by putting the $x-$axis along the direction of propagation and letting the whole problem be one-dimensional, so that $u=u(x,t)$. (We also lose two factors of $\sqrt{2\pi}$.)

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp\left[ikx-i\omega(k)t\right]\,dk \tag{1}$$

Evaluating at $t=0$, we get

$$u(x,0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp(ikx)\,dk$$

and it is tempting to invert this to get the Fourier amplitude $A(k)$ (see 5 below). But there is a second initial condition: $\left.\frac{\partial u}{\partial t}\right|_{t=0}$. So we have to take

$$\begin{aligned} u(x,t) &= \operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp\left[ikx-i\omega(k)t\right]\,dk \\ &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left\{A(k)\exp\left[ikx-i\omega(k)t\right]+A(k)^*\exp\left[i\omega(k)t-ikx\right]\right\}dk \end{aligned} \tag{2}$$

[1] Effectively, $A\left(\vec{k},\omega\right)=A\left(\vec{k}\right)\delta\left[\omega-\omega\left(\vec{k}\right)\right]$, as in Lea Ch 7.

Now we let $\kappa = -k$ in the second term.

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi}\left\{\int_{-\infty}^{+\infty} A(k)\exp\left[ikx - i\omega(k)t\right]\,dk - \int_{+\infty}^{-\infty} A(-\kappa)^*\exp\left[i\kappa x + i\omega(-\kappa)t\right]\,d\kappa\right\} \\ &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left\{A(k)\exp\left[ikx - i\omega(k)t\right] + A(-k)^*\exp\left[ikx + i\omega(k)t\right]\right\}\,dk \qquad (3)\end{aligned}$$

ω is an even function of k because the dispersion relation does not depend on the forward or backward direction of propagation. Then, evaluating at $t = 0$, we get

$$u(\vec{x},0) = \frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[A(\vec{k}) + A(-\vec{k})^*\right]\exp(ikx)\,dk$$

Inverting the transform:

$$\frac{1}{2\sqrt{2\pi}}\left[A(k) + A(-k)^*\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u(x,0)\exp(-ikx)\,dx \qquad (4)$$

If $A(-k) = A(k)^*$, then

$$A(k) = \int_{-\infty}^{+\infty} u(x,0)\exp(-ikx)\,dx \qquad (5)$$

which we would have obtained by the naive approach. To make use of the second initial condition, we take the time derivative of (3).

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left\{-i\omega A(k)\exp\left[ikx - i\omega(k)t\right] + A(-k)^*\, i\omega\exp(-ikx + i\omega t)\right\}\,dk \\ \frac{\partial}{\partial t}u(x,0) &= \frac{1}{4\pi}\int_{-\infty}^{+\infty} -i\omega\left[A(k) - A(-k)^*\right]\exp(ikx)\,dk \end{aligned}$$

Inverting, we have:

$$-\frac{i\omega}{2}\left[A(k) - A(-k)^*\right] = \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}u(x,0)\exp(-ikx)\,dx \qquad (6)$$

Combining the two relations (4) and (6) to eliminate $A(-k)^*$, we have

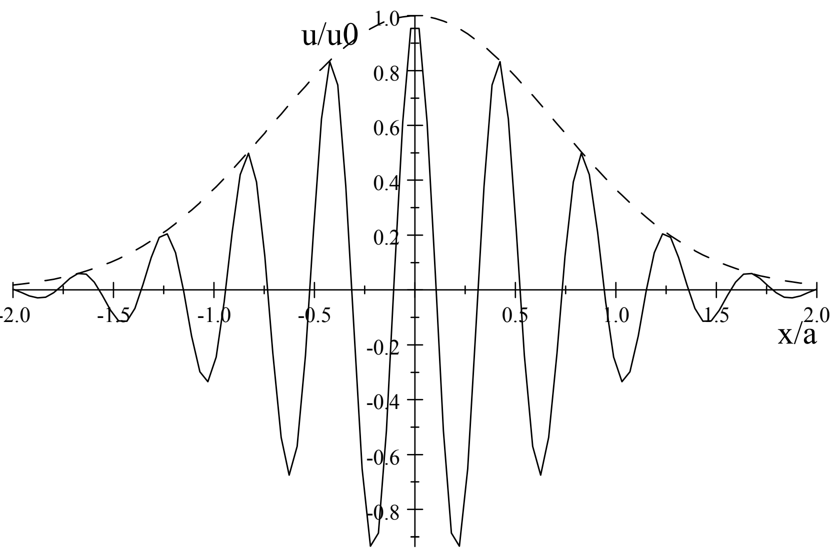
$$A(k) = \int_{-\infty}^{+\infty}\left(u(x,0) - \frac{1}{i\omega}\frac{\partial}{\partial t}u(x,0)\right)\exp(-ikx)\,dx \qquad (7)$$

which is Jackson equation 7.91 modulo a factor of $\sqrt{2\pi}$. (This arises because I started with the transform over space and time, whereas J used the transform over space alone.) We regain equation (5) if $\partial u/\partial t = 0$ at $t = 0$.

2 Gaussian Pulse

To simplify the algebra in this section, let's assume $\frac{\partial}{\partial t}u(x{,}0) = 0$. Generally the wave is sent at a "carrier frequency" ω_0 with a corresponding k_0 so that $\omega_0 = \omega(k_0)$, and the Fourier amplitude $A(k)$ peaks at k_0. For example, a Gaussian pulse with a carrier frequency ω_0 is written:

$$u(x, 0) = u_0 \cos(k_0 x) \exp\left(-\frac{x^2}{a^2}\right) \tag{8}$$



Its transform is:

$$\begin{aligned} A(k) &= \int_{-\infty}^{+\infty} u_0 \cos(k_0 x) \exp\left(-\frac{x^2}{a^2}\right) \exp(-ikx)\, dx \\ &= \int_{-\infty}^{+\infty} \frac{u_0}{2} \left(e^{ik_0 x} + e^{-ik_0 x}\right) \exp\left(-\frac{x^2}{a^2}\right) \exp(-ikx)\, dx \\ &= \frac{u_0}{2} \int_{-\infty}^{+\infty} \left[e^{-i(k-k_0)x} + e^{-i(k_0+k)x}\right] \exp\left(-\frac{x^2}{a^2}\right) dx \end{aligned} \tag{9}$$

So we have two integrals of identical form. To do each, we complete the square (see Lea Example 7.2):

$$\begin{aligned} -\frac{x^2}{a^2} - i(k-k_0)x &= -\frac{1}{a^2}\left[x^2 + i(k-k_0)a^2 x + \left(i\frac{(k-k_0)a^2}{2}\right)^2 - \left(i\frac{(k-k_0)a^2}{2}\right)^2\right] \\ &= -\frac{1}{a^2}\left[\left(i\frac{a^2(k-k_0)}{2} + x\right)^2 + \frac{1}{4}a^4(k-k_0)^2\right] \end{aligned}$$

Thus the first term in the integral (9) is:

$$\begin{aligned}
& \frac{u_0}{2}\exp\left(-\frac{a^2}{4}(k-k_0)^2\right)\int_{-\infty}^{+\infty}\exp\left[-\frac{1}{a^2}\left(i\frac{a^2(k-k_0)}{2}+x\right)^2\right]dx \\
= & \frac{u_0}{2}\exp\left(-\frac{a^2}{4}(k-k_0)^2\right)a\int_{-\infty+i\gamma}^{+\infty+i\gamma}\exp\left(-v^2\right)dv \\
= & \frac{u_0}{2}\exp\left(-\frac{a^2}{4}(k-k_0)^2\right)a\sqrt{\pi} \\
= & \frac{\sqrt{\pi}}{2}au_0\exp\left(-\frac{a^2}{4}(k-k_0)^2\right)
\end{aligned}$$

where we set $v=\left(i\frac{a^2(k-k_0)}{2}+x\right)\frac{1}{a}=i\gamma+x/a$,and used the fact that the integral of the Gaussian is independent of the path between $\pm\infty$. (See Lea Ch 7 pg 328.) To obtain the second term we replace k_0 with $-k_0$. Thus the result for $A(k)$ is two Gaussians, centered at $k=\pm k_0$, and each of width $2/a$.

$$A(k)=A_1(k)+A_2(k)=\sqrt{\pi}a\frac{u_0}{2}\left[\exp\left(-\frac{a^2}{4}(k-k_0)^2\right)+\exp\left(-\frac{a^2}{4}(k+k_0)^2\right)\right] \tag{10}$$

3 Group velocity

For a Gaussian pulse, the integral in (1) has two terms. To evauate the first term, we write $\omega(k)$ in a Taylor series centered at k_0.

$$\omega(k)=\omega_0+(k-k_0)\left.\frac{d\omega}{dk}\right|_{k_0}+\frac{1}{2}(k-k_0)^2\left.\frac{d^2\omega}{dk^2}\right|_{k_0}+\dots$$

Substituting $\omega(k)$ into the expression for $u(x,t)$ (1),

$$u_1=\int_{-\infty}^{+\infty}\frac{A_1(k)}{2\pi}\exp\left\{ikx-it\left[\omega_0+(k-k_0)\left.\frac{d\omega}{dk}\right|_{k_0}+\overset{\text{drop}}{\frac{(k-k_0)^2}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0}}+..\right]\right\}dk \tag{11}$$

and dropping the term in ω'', we have:

$$u_1(x,t)\simeq\frac{1}{2\pi}\exp\left[-i\left(\omega_0-k_0\left.\frac{d\omega}{dk}\right|_{k_0}\right)t\right]\int_{-\infty}^{+\infty}A_1(k)\exp\left(ikx-ik\left.\frac{d\omega}{dk}\right|_{k_0}t\right)dk+\cdots$$

$$\begin{aligned}
u_1(x,t) &= \exp\left[-i\left(\omega_0-k_0\left.\frac{d\omega}{dk}\right|_{k_0}\right)t\right]\frac{1}{2\pi}\int_{-\infty}^{+\infty}A_1(k)\exp\left[ik\left(x-\left.\frac{d\omega}{dk}\right|_{k_0}t\right)\right]dk \\
&= \exp\left[-i\left(\omega_0-k_0\left.\frac{d\omega}{dk}\right|_{k_0}\right)t\right]\times u_1\left(x-\left.\frac{d\omega}{dk}\right|_{k_0}t,0\right) \qquad (12)
\end{aligned}$$

where we used eqn (1) again in the last step, with $x\to x-\left.\frac{d\omega}{dk}\right|_{k_0}t$ and $t\to 0.$. Thus to first order in the expansion of $\omega(k)$, $u(x,t)$ equals a phase factor $e^{-i\phi}$ times $u(x - v_g t, 0)$, that is, at time t the signal looks like the pulse at $t = 0$ translated at speed v_g, where

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0} \tag{13}$$

is the group speed. The phase $\phi = (\omega_0 - v_g k_0)\, t$ indicates that the carrier wave shifts within the Gaussian envelope.

To evaluate the second integral we expand $\omega(k)$ in a Taylor series centered at $-k_0$, and obtain the same result.

4 Pulse spreading

For some waves with $d^2\omega/dk^2 = 0$, there is only a first order term in the Taylor series for $\omega(k)$. But for other waves the higher order terms produce corrections to the first order result. Generally these terms lead to both spreading and distortion of the pulse shape. For example, consider the Whistler (Plasmawaves notes eqn 34):

$$k = \frac{\omega_p}{c}\sqrt{\frac{\omega}{\Omega}}$$

or

$$\omega = \Omega\frac{k^2c^2}{\omega_p^2}$$

The group velocity for this wave is:

$$v_g = \frac{d\omega}{dk} = 2\Omega\frac{kc^2}{\omega_p^2} \tag{14}$$

and the second derivative is

$$v_g' = \frac{d^2\omega}{dk^2} = 2\Omega\frac{c^2}{\omega_p^2} \tag{15}$$

All further derivatives are zero. Thus from (11), the exact expression for u in this case is:

$$u(x,t) = \exp\left[-i\left(\omega_0 - k_0 \left.\frac{d\omega}{dk}\right|_{k_0}\right)t\right] \times \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(k)\exp\left[ik\left(x - \left.\frac{d\omega}{dk}\right|_{k_0} t\right) - i\frac{(k-k_0)^2}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0} t\right] dk$$

$$\begin{aligned} u(x,t) &= \exp\left[-i\left(\omega_0 - k_0 \left.\frac{d\omega}{dk}\right|_{k_0}\right)t\right] \times \\ &\quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(k)\exp\left[ik(x - v_g t) - \frac{i}{2}\left(k^2 - 2kk_0 + k_0^2\right)v_g' t\right] dk \\ &= \exp\left[-i\left(\omega_0 - k_0 v_g + \frac{k_0^2 v_g'}{2}\right)t\right]\frac{1}{2\pi}\int_{-\infty}^{+\infty} A(k)\exp\left[ik\left(x + k_0 v_g' t - v_g t\right) - \frac{i}{2}k^2 v_g' t\right] dk \end{aligned}$$

where v_g and v_g' are given by equations (14) and (15) with $k = k_0$.

Now if we use the Gaussian pulse (10) as an example, we get:

$$u(x,t) = \exp\left[-i\left(\omega_0 - k_0 v_g + \frac{k_0^2 v_g'}{2}\right)t\right]\frac{1}{2\pi}I \tag{16}$$

where

$$I = \sqrt{\pi}\int_{-\infty}^{+\infty}\frac{au_0}{2}\left[\begin{array}{c}\exp\left(-\frac{a^2}{4}(k-k_0)^2\right)\\ +\exp\left(-\frac{a^2}{4}(k+k_0)^2\right)\end{array}\right]\exp\left[ik\left(x \pm k_0 v_g' t - v_g t\right) - \frac{i}{2}k^2 v_g' t\right]dk$$

Again there are two terms, differing only in the sign of k_0 in the Gaussian, and we evaluate these using the Taylor series about $\pm k_0$.

$$I = \frac{\sqrt{\pi}}{2}au_0\left(I_+ + I_-\right)$$

Looking at the first term:

$$\begin{aligned}
I_+ &= \int_{-\infty}^{+\infty}\exp\left[-\frac{a^2}{4}\left(k^2 - 2kk_0 + k_0^2\right) + ik\left(x + k_0 v_g' t - v_g t\right) - \frac{i}{2}k^2 v_g' t\right]dk \\
&= \exp\left(-\frac{a^2 k_0^2}{4}\right)\int_{-\infty}^{+\infty}\exp\left\{-k^2\left(\frac{a^2}{4} + \frac{i}{2}v_g' t\right) + k\left[i\left(x + k_0 v_g' t - v_g t\right) + \frac{k_0 a^2}{2}\right]\right\}dk
\end{aligned}$$

To simplify the notation, we let $s = x - v_g t$ and $v_g' t = \alpha a^2$. Then the argument of the exponential in the integrand is:

$$\begin{aligned}
&-k^2\left(\frac{a^2}{4} + \frac{i}{2}\alpha a^2\right) + k\left[i\left(s + k_0\alpha a^2\right) + \frac{k_0 a^2}{2}\right] \\
= &-\frac{a^2}{4}(1 + 2i\alpha)\left\{k^2 + 2k\left[k_0 + \frac{2is}{a^2(1+2i\alpha)}\right]\right\}
\end{aligned}$$

Completing the square, the term in curly brackets is

$$\begin{aligned}
&k^2 + 2k\left[\frac{2is}{a^2(1+2i\alpha)} + k_0\right] + \left[\frac{2is}{a^2(1+2i\alpha)} + k_0\right]^2 - \left[\frac{2is}{a^2(1+2i\alpha)} + k_0\right]^2 \\
= &\left[k + \frac{2is}{a^2(1+2i\alpha)} + k_0\right]^2 - \left[\frac{2is}{a^2(1+2i\alpha)} + k_0\right]^2
\end{aligned}$$

We can do the integral by making the change of variable:

$$\xi = \frac{a}{2}\sqrt{1+2i\alpha}\left[k + \frac{2is}{a^2(1+2i\alpha)} + k_0\right]$$

Then:

$$I_+ = \exp\left(-\frac{a^2 k_0^2}{4}\right)\exp\left\{\frac{a^2}{4}(1+2i\alpha)\left[\frac{2is}{a^2(1+2i\alpha)} + k_0\right]^2\right\}\frac{2}{a\sqrt{1+2i\alpha}}\int_{-\infty+i\gamma}^{+\infty+i\gamma}e^{-\xi^2}d\xi$$

where the path of integration is again moved off the real axis, but the result is still $\sqrt{\pi}$. Thus:

$$I_+ = \frac{2\sqrt{\pi}}{a\sqrt{1+2i\alpha}}\exp\left\{-\frac{a^2 k_0^2}{4} - \frac{s^2}{a^2(1+2i\alpha)} + isk_0 + \frac{k_0^2 a^2}{4}(1+2i\alpha)\right\} \tag{17}$$

$$= \frac{2\sqrt{\pi}}{a\sqrt{1+2i\alpha}}\exp\left\{-\frac{s^2(1-2i\alpha)}{a^2(1+4\alpha^2)} + isk_0 + \frac{i}{2}\alpha k_0^2 a^2\right\} \tag{18}$$

And the argument of the exponential in u (16) is:

$$\begin{aligned} & i\left(-\omega_0 t + k_0 v_g t - \frac{k_0^2 a^2 \alpha}{2} + k_0 s + \frac{\alpha k_0^2 a^2}{2}\right) - \frac{s^2\left(1-2i\alpha\right)}{a^2\left(1+4\alpha^2\right)} \\ = & \; i\left(-\omega_0 t + k_0 x\right) - \frac{s^2\left(1-2i\alpha\right)}{a^2\left(1+4\alpha^2\right)} \\ = & \; i\left(k_0 x - \omega_0 t\right) - \frac{s^2}{a^2\left(1+4\alpha^2\right)} + \frac{2i\alpha s^2}{a^2\left(1+4\alpha^2\right)} \end{aligned}$$

Thus

$$\begin{aligned} u\left(x,t\right) &= \sqrt{\pi}\frac{a u_0}{2}\frac{1}{2\pi}\exp\left[i\left(k_0 x - \omega_0 t\right)\right]\exp\left[-\frac{s^2}{a^2\left(1+4\alpha^2\right)}\right] \times \\ & \quad \exp\left[i\alpha\frac{2s^2}{a^2\left(1+4\alpha^2\right)}\right]\frac{2\sqrt{\pi}}{a\sqrt{1+2i\alpha}} + \left(k_0 \to -k_0\right) \\ &= \frac{u_0}{2}\frac{\exp\left[i\left(k_0 x - \omega_0 t\right)\right]}{\left(1+4\alpha^2\right)^{1/4}}\exp\left(-\frac{\left(x - v_g t\right)^2}{a^2\left(1+4\alpha^2\right)}\right)e^{-i\phi} + \left(k_0 \to -k_0\right) \quad (19) \end{aligned}$$

We may interpret this expression as folows:

- Initial wave form at frequency ω_0. This is the carrier.
- Envelope peaks at $x = v_g t$ due to its travelling at the group speed.
- Envelope spread by factor $\sqrt{1+4\alpha^2}$ where $\alpha = v_g' t/a^2$
- Amplitude decreased by a similar factor $\left(1+4\alpha^2\right)^{1/4}$.
- Overall phase factor

$$\phi = -\alpha\frac{2s^2}{a^2\left(1+4\alpha^2\right)} + \frac{\tan^{-1}\left(2\alpha\right)}{2}$$

approximately proportional to $\alpha \propto t$.

At large times the pulse width grows roughly linearly with time. $a\left(t\right) \simeq 2a\left(0\right)\alpha = 2v_g' t/a\left(0\right)$. Notice that (19) gives the correct result (8) at $t = 0$.

5 Arrival of a signal

The previous analysis shows the shape of the signal over all space as a function of time. But this is not usually what we observe. Let us now consider a signal generated at $x = 0$ over a period of time. We want to see what kind of a signal arrives at a distant point $x = X$ as a function of time. So instead of using the dispersion relation in the form $\omega\left(k\right)$ we instead think of it as $k\left(\omega\right)$. Then:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} A(\omega)\exp\left[ik\left(\omega\right)x - i\omega t\right]\, d\omega$$

where in general $k(\omega)$ is complex.

As we have seen, $A\left(\omega\right)$ is usually a smooth function, somewhat peaked around the carrier frequency ω_0. The exponential oscillates, so it tends to make the value of the integral small. However, when the exponent stays almost constant over a range of ω, we will have a subtantial contribution to the integral. This happens when the phase is *stationary.* (See Lea Optional Topic D section 2)

$$\begin{aligned}\frac{d}{d\omega}\left[k\left(\omega\right)x-\omega t\right] &= 0\\ x\frac{dk}{d\omega}-t &= 0\end{aligned}$$

or

$$x=\frac{d\omega}{dk}t=v_g t \tag{20}$$

Thus the major contribution to the integral is from the frequency that, at time t, has its group speed equal to x/t. Put another way, the signal at that frequency, travelling at the group speed, has just reached the observation point.

To find the received signal, we must evaluate the integral. Again we expand $k\left(\omega\right)$ in a Taylor series, this time about the stationary frequency. as determined by equation (20).

$$k\left(\omega\right)=k_s+\left(\omega-\omega_s\right)\left.\frac{dk}{d\omega}\right|_{\omega_s}+\frac{1}{2}\left(\omega-\omega_s\right)^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}+\cdots$$

where $k_s=k\left(\omega_s\right)$, giving

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}A(\omega)\exp\left\{i\left[k_s+\left(\omega-\omega_s\right)\left.\frac{dk}{d\omega}\right|_{\omega_s}+\frac{\left(\omega-\omega_s\right)^2}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}+\cdots\right]x-i\omega t\right\}d\omega$$

Now use the stationary phase condition (20):

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}A(\omega)\exp\left\{i\left[k_s-\frac{\omega_s}{v_g}+\frac{1}{2}\left(\omega-\omega_s\right)^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}+..\right]v_g t\right\}d\omega$$

Since the exponential guarantees that only frequencies near ω_s contribute, we may pull out the slowly varying amplitude, as well as the terms in the exponential that are independent of ω:

$$u(x,t)\simeq\frac{A(\omega_s)}{\sqrt{2\pi}}\exp\left[i\left(k_s v_g-\omega_s\right)t\right]\int_{-\infty}^{+\infty}\exp\left[\frac{i}{2}\left(\omega-\omega_s\right)^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}v_g t\right]d\omega$$

To do the integral, change variables to:

$$\xi=\left(\omega-\omega_s\right)\sqrt{\frac{-i}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}v_g t}=e^{-i\pi/4}\left(\omega-\omega_s\right)\sqrt{\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}\frac{v_g t}{2}}$$

Then

$$\begin{aligned}I &= \int_{-\infty}^{+\infty}\exp\left[\frac{i}{2}\left(\omega-\omega_s\right)^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}v_g t\right]d\omega=\frac{e^{i\pi/4}}{\sqrt{\frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}v_g t}}\int_{-\infty+i\gamma}^{+\infty+i\gamma}e^{-\xi^2}d\xi\\ &= \sqrt{\frac{2\pi}{\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s}v_g t}}e^{i\pi/4}\end{aligned}$$

and so, incorporating the result (20), we have

$$u(x,t) = \frac{A(\omega_s)}{\sqrt{\frac{d^2k}{d\omega^2}\Big|_{\omega_s} x}} \exp\left[i\left(k_s x - \omega_s t\right) + i\frac{\pi}{4}\right] \tag{21}$$

Again we see the carrier wave, modulated by an envelope that changes in time, and with an additional phase change. (Remember that ω_s is a function of time.)

To see how this works out, again look at the whistler. The stationary phase condition is (equations 14 and 20)

$$\begin{aligned} x &= 2\Omega \frac{kc^2}{\omega_p^2} t = \frac{\omega_p}{c}\sqrt{\frac{\omega}{\Omega}} 2\Omega \frac{c^2}{\omega_p^2} t \\ &= 2\frac{\sqrt{\omega\Omega}}{\omega_p} ct \end{aligned}$$

and thus

$$\omega_s = \frac{\omega_p^2}{\Omega}\frac{x^2}{4c^2t^2}$$

which decreases from a large value toward zero as t increases. For this wave the signal (21) is given by:

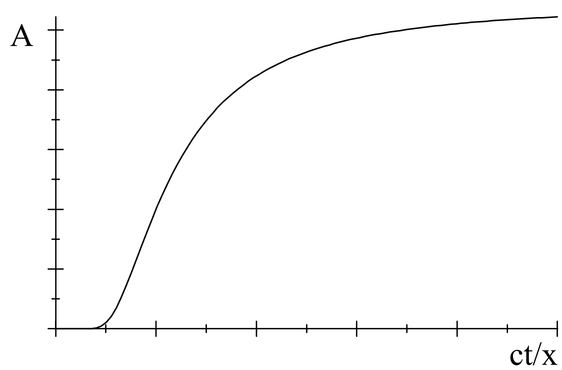
$$\begin{aligned} u(x,t) &= \sqrt{2\Omega\frac{c^2}{\omega_p^2}}\frac{A(\frac{\omega_p^2}{\Omega}\frac{x^2}{4c^2t^2})}{\sqrt{x}} \exp\left[i\left(k_s v_g - \omega_s\right)t + i\frac{\pi}{4}\right] \\ &= \frac{c}{\omega_p}\sqrt{\frac{2\Omega}{x}} A(\frac{\omega_p^2}{\Omega}\frac{x^2}{4c^2t^2}) \exp\left[i\left(k_s v_g - \omega_s\right)t + i\frac{\pi}{4}\right] \end{aligned}$$

If A is a Gaussian as in (10), then

$$u(x,t) = \frac{c}{\omega_p}\sqrt{\frac{2\Omega}{x}} A_0 \exp\left[-\eta\left(\frac{\omega_p^2}{\Omega}\frac{x^2}{4c^2t^2} - \omega_0\right)^2\right] \exp\left[i\left(k_s v_g - \omega_s\right)t + i\frac{\pi}{4}\right]$$

where η is a constant.

At fixed x, we get something like this:



The signal grows from zero to an asymptotic value, until the analysis breaks down at low frequency (large t). (Remember: we neglected ion motion in getting the dispersion relation.)

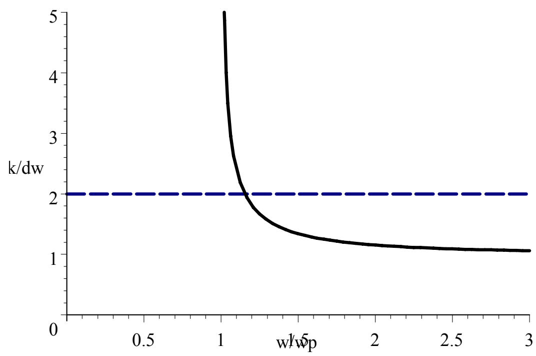
Here's another way to look at it. Consider the plasma waves (Plasmawave notes eqn 12):

$$c^2k^2 = \omega^2 - \omega_p^2$$

For this wave

$$\begin{aligned}
2c^2k\frac{dk}{d\omega} &= 2\omega \\
c\frac{dk}{d\omega} &= c\frac{\omega}{kc^2} = \frac{\omega}{\sqrt{\omega^2 - \omega_p^2}} = \frac{\omega/\omega_p}{\sqrt{(\omega/\omega_p)^2 - 1}}
\end{aligned}$$

which looks like this:



The stationary phase condition is

$$\frac{ct}{x} = c\frac{dk}{d\omega}$$

There is no signal at a fixed $x = x_0$ prior to $t = t_0 = x_0/c$. The signal begins at infinite frequency, and moves to lower frequency as time increases, asymptotically reaching ω_p. The horizontal line in the graph represents $ct/x = 2$ and the intersection of the two lines is the observed frequency at this time.

For a more complicated dispersion relation, the signal can be more complicated. Consider the RHC wave propagated along $\vec{B}$. (Plasmawave notes eqn 28 with the minus sign.) Then:

$$\begin{aligned}
c^2k^2 &= \omega^2 - \frac{\omega_p^2\omega}{\omega - \Omega} \\
2c^2k\frac{dk}{d\omega} &= 2\omega - \frac{\omega_p^2}{\omega - \Omega} + \frac{\omega_p^2\omega}{(\omega - \Omega)^2} \\
&= 2\omega - \frac{\omega_p^2}{\omega - \Omega}\left(1 - \frac{\omega}{\omega - \Omega}\right) \\
&= 2\omega + \frac{\omega_p^2\Omega}{(\omega - \Omega)^2}
\end{aligned}$$

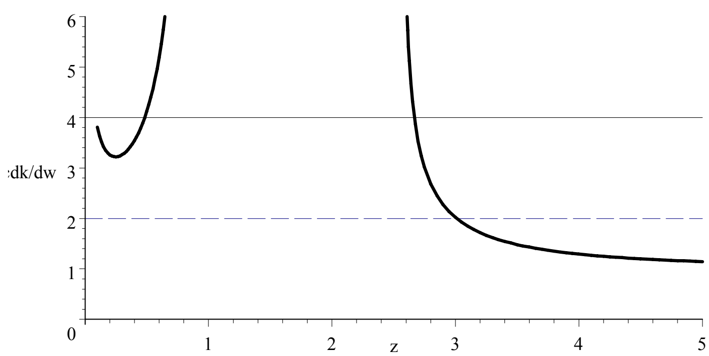
and thus:

$$
\begin{aligned}
c\frac{dk}{d\omega} &= \frac{1}{ck}\left(\omega + \frac{1}{2}\frac{\omega_p^2\Omega}{(\omega-\Omega)^2}\right) \\
&= \frac{1}{\sqrt{\omega^2 - \frac{\omega_p^2\omega}{\omega-\Omega}}}\left[\omega + \frac{1}{2}\frac{\omega_p^2\Omega}{(\omega-\Omega)^2}\right] \\
&= \frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega(\omega-\Omega)}}}\left[1 + \frac{1}{2}\frac{\omega_p^2\Omega/\omega}{(\omega-\Omega)^2}\right]
\end{aligned}
$$

Now let $\omega_p/\Omega = y$ and $\omega/\Omega = z$ Then

$$
c\frac{dk}{d\omega} = \frac{1}{\sqrt{1 - \frac{y^2}{z(z-1)}}}\left(1 + \frac{1}{2}\frac{y^2}{z(z-1)^2}\right)
$$

With $y = 2$, the diagram looks like:



As t increases we first obtain a signal at one frequency (dashed line) and later at 3 frequencies, two decreasing and one increasing (solid line) .