EM Waves in vacuum

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1 Polarization

Here we want to investigate the vector nature of the E&M fields in a wave. As usual, Maxwell's equations tell the whole tale. In a source-free region:

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

If we now assume that each field has the plane wave form $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, then we find:

$$\vec{k} \cdot \vec{E} = 0$$
$$\vec{k} \cdot \vec{B} = 0$$

i.e. both $\vec{E}$ and $\vec{B}$ are perpendicular to $\vec{k}$, and:

$$\vec{k} \times \vec{E} = \omega \vec{B} \tag{1}$$

so $\vec{B}$ is also perpendicular to $\vec{E}$. Now if we put the $z-$axis along $\vec{k}$, $\vec{k} = k\hat{z}$, then we may express the wave amplitudes as:

$$\vec{E}_0 = E_1\hat{x} + E_2\hat{y}$$

and similarly for $\vec{B}_0$. Finally we may allow for phase shifts by letting $E_j = E_{j0}e^{i\phi_j}$. Since $\vec{E}$ and $\vec{B}$ are perpendicular, and have related magnitudes (by eqn (1), then

$$B_2 = \frac{k}{\omega}E_1 \text{ and } B_1 = -\frac{k}{\omega}E_2$$

so we can focus attention on the components of $\vec{E}_0$.

1.1 Linear polarization

Remember that the real physical field is the real part of the complex number. If $\phi_1 = \phi_2$, then both components of $\vec{E}$ vary in phase, and $E_1/E_2 = E_{10}/E_{20}$ is constant in time and space. Thus the electric field vector has a constant direction as its magnitude varies. The wave is *linearly polarized*.

1.2 Elliptical polarization

If $\phi_2 = \phi_1 + \frac{\pi}{2}$, then at $z = 0$ we find:

$$\begin{aligned}
\vec{E} &= \text{Re}(E_{10}e^{i(\phi_1-\omega t)}\hat{x} + E_{20}e^{i(\phi_2-\omega t)}\hat{y}) \qquad (2)\\
&= \text{Re}(E_{10}e^{i(\phi_1-\omega t)}\hat{x} + E_{20}e^{i\left(\phi_1+\frac{\pi}{2}-\omega t\right)}\hat{y}) \qquad \text{if } \phi_2 = \phi_1 + \pi/2\\
&= E_{10}\cos(\phi_1 - \omega t)\,\mathbf{\hat{x}} - E_{20}\sin(\phi_1 - \omega t)\,\mathbf{\hat{y}} = E_{10}\cos(\omega t - \phi_1)\,\mathbf{\hat{x}} + E_{20}\sin(\omega t - \phi_1)\,\mathbf{\hat{y}}
\end{aligned}$$

As t increases from ϕ_1/ω, E_x decreases and E_y increases: The electric field vector rotates counter-clockwise, and the tip of the vector traces out an ellipse as the vector rotates. This is an *elliptically polarized* wave. When the amplitudes E_{10} and E_{20} are equal, the curve is a circle and we have a *right-hand circularly polarized* wave.. Stick the thumb of your right hand in the direction of propagation (the $z-$direction in this case) and your fingers curl in the direction that $\vec{E}$ rotates. If instead we take $\phi_2 = \phi_1 - \frac{\pi}{2}$, then we get:

$$\vec{E} = E_{10}\cos(\omega t - \phi_1)\ \hat{x} - E_{20}\sin(\omega t - \phi_1)\ \hat{y}$$

and the $y-$component becomes increasingly negative: *i.e.* the vector rotates clockwise. When the amplitudes E_{10} and E_{20} are equal, the curve is a circle and we have *left-hand circular polarization*.

1.3 Circular polarization

In the case of circular polarization, with $E_{10} = E_{20} = E_0/\sqrt{2}$, we have

$$\vec{E} = E_0\left(\frac{\hat{x} \pm i\hat{y}}{\sqrt{2}}\right)e^{i(\phi_1-\omega t)} \qquad (3)$$

so it is convenient to use the complex polarization vectors

$$\hat{e}_{\text{R,L}} = \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}} \qquad (4)$$

which have the orthogonality properties

$$\hat{e}_{\text{R}} \cdot \hat{e}_{\text{L}}^* = 0 \quad \text{and } \hat{e}_{\text{R}} \cdot \hat{e}_{\text{R}}^* = \hat{e}_{\text{L}} \cdot \hat{e}_{\text{L}}^* = 1$$

1.4 General case

Any wave may be decomposed into a sum of linearly polarized waves or a sum of circularly polarized waves. In the most general case we have elliptical polarization with the axes of the ellipse oriented at an angle θ to the x and y axes, where θ is unknown for the moment. Given E_{10}, $E_{20,}$ ϕ_1 and ϕ_2, we'd like to find the shape and orientation of the ellipse. It's most convenient to do the analysis using the circular polarization vectors (4). Let's rewrite

$\vec{E}$ (eqn 2) in terms of $\hat{e}_\text{R}$ and $\hat{e}_\text{L}$:

$$\begin{aligned}
\vec{E} &= E_{10}e^{i(\phi_1-\omega t)}\hat{x} + E_{20}e^{i(\phi_2-\omega t)}\hat{y} \\
&= E_{10}e^{i(\phi_1-\omega t)}\left(\frac{\hat{e}_\text{R}+\hat{e}_\text{L}}{\sqrt{2}}\right) + E_{20}e^{i(\phi_2-\omega t)}\left(\frac{\hat{e}_\text{R}-\hat{e}_\text{L}}{\sqrt{2}i}\right) \\
&= \left[\left(E_{10}e^{i\phi_1} - iE_{20}e^{i\phi_2}\right)\left(\frac{\hat{e}_\text{R}}{\sqrt{2}}\right) + \left(E_{10}e^{i\phi_1} + iE_{20}e^{i\phi_2}\right)\left(\frac{\hat{e}_\text{L}}{\sqrt{2}}\right)\right] e^{-i\omega t} \\
&= \left(E_\text{R}\hat{e}_\text{R} + E_\text{L}\hat{e}_\text{L}\right) e^{-i\omega t} \qquad (5)
\end{aligned}$$

where

$$E_\text{R} = \frac{1}{\sqrt{2}}\left[E_{10}\cos\phi_1 + E_{20}\sin\phi_2 + i\left(E_{10}\sin\phi_1 - E_{20}\cos\phi_2\right)\right] = \left|E_\text{R}\right| e^{i\chi_\text{R}}$$

The magnitude is given by

$$\begin{aligned}
\left|E_\text{R}\right| &= \frac{1}{\sqrt{2}}\sqrt{\left(E_{10}\cos\phi_1 + E_{20}\sin\phi_2\right)^2 + \left(E_{10}\sin\phi_1 - E_{20}\cos\phi_2\right)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\sin\left(\phi_2-\phi_1\right)} \qquad (6)
\end{aligned}$$

and the phase by

$$\tan\chi_\text{R} = \frac{E_{10}\sin\phi_1 - E_{20}\cos\phi_2}{E_{10}\cos\phi_1 + E_{20}\sin\phi_2} \qquad (7)$$

For the left-hand component, we have

$$E_\text{L} = \frac{1}{\sqrt{2}}\left[E_{10}\cos\phi_1 - E_{20}\sin\phi_2 + i\left(E_{10}\sin\phi_1 + E_{20}\cos\phi_2\right)\right] = \left|E_\text{L}\right| e^{i\chi_\text{L}}$$

with

$$\left|E_\text{L}\right| = \frac{1}{\sqrt{2}}\sqrt{E_{10}^2 + E_{20}^2 - 2E_{10}E_{20}\sin\left(\phi_2-\phi_1\right)} \qquad (8)$$

and

$$\tan\chi_\text{L} = \frac{E_{10}\sin\phi_1 + E_{20}\cos\phi_2}{E_{10}\cos\phi_1 - E_{20}\sin\phi_2} \qquad (9)$$

If $\phi_2 - \phi_1 = \pi/2$, then $\left|E_\text{R}\right| = E_{10} + E_{20}$ and $\left|E_\text{L}\right| = \left|E_{10} - E_{20}\right|$. Additionally, if $E_{10} = E_{20}$, $\left|E_\text{L}\right| = 0$ as expected for right circular polarization.

We may factor (5) to get:

$$\begin{aligned}
\vec{E} &= \left|E_\text{R}\right|\left[\hat{e}_\text{R} + \hat{e}_\text{L}\frac{\left|E_\text{L}\right|}{\left|E_\text{R}\right|}\exp i\left(\chi_\text{L}-\chi_\text{R}\right)\right] e^{i(\chi_\text{R}-\omega t)} \\
&= \left|E_\text{R}\right|\left[\hat{e}_\text{R} + \varepsilon\hat{e}_\text{L}\exp i\alpha\right] e^{i(\chi_\text{R}-\omega t)} \qquad (10)
\end{aligned}$$

where the last line defines ε and α. In particular, using

$$\tan\left(A+B\right) = \frac{\tan A + \tan B}{1 - \tan A\tan B},$$

along with (7) and (9), we have

$$\begin{aligned}
\alpha &= \chi_{\rm L}-\chi_{\rm R}=\tan^{-1}\frac{E_{10}\sin\phi_1+E_{20}\cos\phi_2}{E_{10}\cos\phi_1-E_{20}\sin\phi_2}-\tan^{-1}\frac{E_{10}\sin\phi_1-E_{20}\cos\phi_2}{E_{10}\cos\phi_1+E_{20}\sin\phi_2}\\
&= \tan^{-1}\frac{\frac{E_{10}\sin\phi_1+E_{20}\cos\phi_2}{E_{10}\cos\phi_1-E_{20}\sin\phi_2}-\frac{E_{10}\sin\phi_1-E_{20}\cos\phi_2}{E_{10}\cos\phi_1+E_{20}\sin\phi_2}}{1+\frac{E_{10}\sin\phi_1+E_{20}\cos\phi_2}{E_{10}\cos\phi_1-E_{20}\sin\phi_2}\frac{E_{10}\sin\phi_1-E_{20}\cos\phi_2}{E_{10}\cos\phi_1+E_{20}\sin\phi_2}}\\
&= \tan^{-1}\frac{2E_{10}E_{20}\cos(\phi_2-\phi_1)}{E_{10}^2\cos^2\phi_1-E_{20}^2\sin^2\phi_2+E_{10}^2\sin^2\phi_1-E_{20}^2\cos^2\phi_2}\\
&= \tan^{-1}\frac{2E_{10}E_{20}\cos(\phi_2-\phi_1)}{E_{10}^2-E_{20}^2} \qquad (11)
\end{aligned}$$

Expanding the circular polarization vectors in (10), we get

$$\vec{E}=\frac{|E_{\rm R}|}{\sqrt{2}}\left[\left(1+\varepsilon e^{i\alpha}\right)\hat{x}+i\left(1-\varepsilon e^{i\alpha}\right)\hat{y}\right]e^{i(\chi_{\rm R}-\omega t)} \qquad (12)$$

The physical field is the real part of (12):

$$\vec{E}_{\rm phys}=\frac{|E_{\rm R}|}{\sqrt{2}}\left\{\left[\cos(\chi_{\rm R}-\omega t)+\varepsilon\cos(\alpha+\chi_{\rm R}-\omega t)\right]\hat{x}-\left[\sin(\chi_{\rm R}-\omega t)-\varepsilon\sin(\alpha+\chi_{\rm R}-\omega t)\right]\hat{y}\right\} \qquad (13)$$

with magnitude

$$\begin{aligned}
\left|\vec{E}_{\rm phys}\right| &= \frac{|E_{\rm R}|}{\sqrt{2}}\sqrt{\left[\cos(\chi_{\rm R}-\omega t)+\varepsilon\cos(\alpha+\chi_{\rm R}-\omega t)\right]^2+\left[\sin(\chi_{\rm R}-\omega t)-\varepsilon\sin(\alpha+\chi_{\rm R}-\omega t)\right]^2}\\
&= \frac{|E_{\rm R}|}{\sqrt{2}}\sqrt{\begin{array}{c}\cos^2(\chi_{\rm R}-\omega t)+\varepsilon^2\cos^2(\alpha+\chi_{\rm R}-\omega t)+2\varepsilon\cos(\chi_{\rm R}-\omega t)\cos(\alpha+\chi_{\rm R}-\omega t)\\ +\sin^2(\chi_{\rm R}-\omega t)+\varepsilon^2\sin^2(\alpha+\chi_{\rm R}-\omega t)-2\varepsilon\sin(\alpha+\chi_{\rm R}-\omega t)\sin(\chi_{\rm R}-\omega t)\end{array}}\\
&= \frac{|E_{\rm R}|}{\sqrt{2}}\sqrt{1+\varepsilon^2+2\varepsilon\cos\left[\alpha+2(\chi_{\rm R}-\omega t)\right]}
\end{aligned}$$

Thus $\left|\vec{E}\right|$ is maximum and equals

$$\frac{|E_{\rm R}|}{\sqrt{2}}(1+\varepsilon)$$

when $\omega t=\alpha/2+\chi_{\rm R}$. At the maximum, $\vec{E}$ makes angle θ with the $x-$axis, where, from (13),

$$\tan\theta=\frac{E_y}{E_x}=\frac{\sin(\alpha/2)+\varepsilon\sin(\alpha/2)}{\cos(\alpha/2)+\varepsilon\cos(\alpha/2)}=\tan\frac{\alpha}{2} \qquad (14)$$

The minimum occurs when

$$\alpha+2(\chi_{\rm R}-\omega t)=\pi$$

and equals

$$\frac{|E_{\rm R}|}{\sqrt{2}}(1-\varepsilon)$$

In this case

$$\tan\theta=\frac{\cos\frac{\alpha}{2}-\varepsilon\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\varepsilon\sin\frac{\alpha}{2}}=\cot\frac{\alpha}{2}$$

So the minor axis is perpendicular to the major axis. Thus we have elliptical polarization with major axis rotated through angle $\alpha/2$ from the x-axis. (See eqn (11). θ is zero if

$|\phi_1 - \phi_2| = \pi/2$, as expected.) The ratio of major to minor axes is

$$\frac{E_{\text{max}}}{E_{\text{min}}} = \frac{|E_{\text{R}}| + |E_{\text{L}}|}{||E_{\text{R}}| - |E_{\text{L}}||} = \frac{1}{\sqrt{1-e^2}} \tag{15}$$

where e is the eccentricity.

1.5 Stokes Parameters

What do we actually observe? We can design our detectors to measure either linear or circular polarizations. From those observations, we would like to determine the polarization characteristics of the observed radiation. That is, we will observe

$$\hat{x} \cdot \vec{E} = E_x = a_1 e^{i\delta_1}, \;\; \hat{y} \cdot \vec{E} = E_y = a_2 e^{i\delta_2}$$

or

$$\hat{e}_{\text{R}} \cdot \vec{E} = E_{\text{R}} = a_R e^{i\delta_R}, \;\;\; \hat{e}_{\text{L}} \cdot \vec{E} = E_{\text{L}} = a_L e^{i\delta_L}$$

(In our previous notation ($E_{10} = a_1$ and $\phi_1 = \delta_1$, $E_{20} = a_2$ and $\phi_2 = \delta_2$. I have now switched to Jackson's notation (pg301).)

The Stokes parameters are defined as:

$$\begin{aligned}
s_0 &= \left|E_x^2\right| + \left|E_y\right|^2 = a_1^2 + a_2^2 = \text{ total intensity} \\
s_1 &= \left|E_x^2\right| - \left|E_y\right|^2 = a_1^2 - a_2^2 \\
s_2 &= 2\,\text{Re}\left(E_x^* E_y\right) = 2a_1 a_2 \cos\left(\delta_2 - \delta_1\right) \\
s_3 &= 2\,\text{Im}\left(E_x^* E_y\right) = 2a_1 a_2 \sin\left(\delta_2 - \delta_1\right)
\end{aligned}$$

with a similar set for the circular polarizations. The four parameters are not independent:

$$s_0^2 = s_1^2 + s_2^2 + s_3^2$$

They describe physical parameters as follows:

$$\frac{s_1}{s_0} = \text{percent linear polarization}$$

From equation (11) we get

$$\frac{s_2}{s_1} = \tan(2 \times \text{angle of polarization ellipse})$$

The eccentricity of the polarization ellipse is given by (equations 15, 6, and 8)

$$\begin{aligned}
e^2 &= 1 - \left(\frac{|E_{\text{R}}| - |E_{\text{L}}|}{||E_{\text{R}}| + |E_{\text{L}}||}\right)^2 = \frac{4a_R a_L}{\left(a_R + a_L\right)^2} \\
&= \frac{4\frac{1}{\sqrt{2}}\sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \sin\left(\phi_2 - \phi_1\right)}\frac{1}{\sqrt{2}}\sqrt{a_1^2 + a_2^2 - 2a_1 a_2 \sin\left(\phi_2 - \phi_1\right)}}{\left[\frac{1}{\sqrt{2}}\sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \sin\left(\phi_2 - \phi_1\right)} + \frac{1}{\sqrt{2}}\sqrt{a_1^2 + a_2^2 - 2a_1 a_2 \sin\left(\phi_2 - \phi_1\right)}\right]^2} \\
&= \frac{4\sqrt{s_0^2 - s_3^2}}{\left[\left[\sqrt{s_0 + s_3} + \sqrt{s_0 - s_3}\right]\right]^2} = \frac{4\sqrt{s_0^2 - s_3^2}}{s_0 + s_3 + (s_0 - s_3) + 2\sqrt{s_0^2 - s_3^2}} \\
&= \frac{2\sqrt{s_0^2 - s_3^2}}{s_0 + \sqrt{s_0^2 - s_3^2}} = \frac{2\sqrt{s_1^2 + s_2^2}}{s_0 + \sqrt{s_1^2 + s_2^2}}
\end{aligned}$$

Check: If $\delta_1 = \delta_2$ then $s_2 = 2a_1 a_2$ and

$$\sqrt{s_1^2 + s_2^2} = \sqrt{\left(a_1^2 - a_2^2\right)^2 + 4a_1^2 a_2^2} = s_0$$

So we get $e = 1$, as expected. The polarization is linear with no circularly polarized component.

See also Pacholczyk, Radio Astrophysics, Appendix I