Perturbation theory for scattering in an almost uniform medium

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Suppose we have a medium with average dielectic constant ε_0 and magnetic permeability μ_0, but with $\varepsilon \neq \varepsilon_0$ and $\mu \neq \mu_0$ in small regions. Note: We are in Gaussian units. ε_0 is NOT the SI ε_0, but the unperturbed value of ε.

Maxwell's equations for this system are:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1}$$

$$\vec{\nabla} \cdot \vec{D} = 0 \tag{2}$$

(no free charge)

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{3}$$

and

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \tag{4}$$

together with

$$\vec{D} = \varepsilon \vec{E} \tag{5}$$

and

$$\vec{B} = \mu \vec{H} \tag{6}$$

We want to separate out those regions which deviate from the uniform properties, so we write:

$$\vec{D} = \vec{D} - \varepsilon_0 \vec{E} + \varepsilon_0 \vec{E}$$

Then

$$\vec{\nabla} \times \left[\vec{\nabla} \times \left(\vec{D} - \varepsilon_0 \vec{E}\right)\right] = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{D}\right) - \varepsilon_0 \vec{\nabla} \times \left(-\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right)$$

where we used equation (3). Thus:

$$\begin{aligned}
\vec{\nabla} \times \left[\vec{\nabla} \times \left(\vec{D} - \varepsilon_0 \vec{E}\right)\right] &= \vec{\nabla}\left(\vec{\nabla} \cdot \vec{D}\right) - \nabla^2 \vec{D} + \frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) \\
&= -\nabla^2 \vec{D} + \frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right)
\end{aligned}$$

where we used equation (2). Now we do the same kind of thing with $\vec{B}$:

$$\vec{B} = \vec{B} - \mu_0 \vec{H} + \mu_0 \vec{H}$$

so

$$\begin{aligned}\vec{\nabla}\times\vec{B} &= \vec{\nabla}\times\left(\vec{B}-\mu_0\vec{H}+\mu_0\vec{H}\right)\\ &= \vec{\nabla}\times\left(\vec{B}-\mu_0\vec{H}\right)+\frac{\mu_0}{c}\frac{\partial\vec{D}}{\partial t}\end{aligned}$$

(equation 4) Thus:

$$\vec{\nabla}\times\left[\vec{\nabla}\times\left(\vec{D}-\varepsilon_0\vec{E}\right)\right] = -\nabla^2\vec{D}+\frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\left[\vec{\nabla}\times\left(\vec{B}-\mu_0\vec{H}\right)+\frac{\mu_0}{c}\frac{\partial\vec{D}}{\partial t}\right]$$

and thus:

$$\nabla^2\vec{D}-\frac{\mu_0\varepsilon_0}{c^2}\frac{\partial^2\vec{D}}{\partial t^2} = -\vec{\nabla}\times\left[\vec{\nabla}\times\left(\vec{D}-\varepsilon_0\vec{E}\right)\right]+\frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\left(\vec{B}-\mu_0\vec{H}\right) \qquad (7)$$

This equation shows that the differences between the local value of $\vec{D}$ and its average value $\vec{D}_{\text{ave}} = \varepsilon_0\vec{E}$, and between the local value of $\vec{B}$ and its average value $\vec{B}_{\text{ave}} = \mu_0\vec{H}$ act as sources for $\vec{D}$.

Now let all quantities have the oscillating form $e^{-i\omega t}$. The equation (7) becomes

$$\left(\nabla^2+k^2\right)\vec{D} = -\vec{\nabla}\times\left[\vec{\nabla}\times\left(\vec{D}-\varepsilon_0\vec{E}\right)\right]-i\frac{\varepsilon_0\omega}{c}\vec{\nabla}\times\left(\vec{B}-\mu_0\vec{H}\right)$$

where

$$k^2 = \frac{\mu_0\varepsilon_0}{c^2}\omega^2 = \frac{\omega^2}{v^2}$$

and $v = c/\sqrt{\mu_0\varepsilon_0}$ is the wave speed in the medium. We can use the Green's function for the wave equation to obtain an expression for $\vec{D}$:

$$\vec{D} = \vec{D}^{(0)}+\frac{1}{4\pi}\int dV\frac{\exp\left(ik\left|\vec{x}-\vec{x}'\right|\right)}{\left|\vec{x}-\vec{x}'\right|}\left\{\vec{\nabla}\times\left[\vec{\nabla}\times\left(\vec{D}-\varepsilon_0\vec{E}\right)\right]+i\frac{\varepsilon_0\omega}{c}\vec{\nabla}\times\left(\vec{B}-\mu_0\vec{H}\right)\right\} \qquad (8)$$

Of course this is not really a solution yet, because $\vec{D}$ appears on the right hand side as well.

Now we expect the solution to be a radiation field of the form

$$\vec{D} = \vec{D}^{(0)}+\vec{A}_{sc}\frac{e^{ikr}}{r}$$

where, from equation 8, the scattering amplitude is:

$$\vec{A}_{sc} = \frac{1}{4\pi}\int dV'\exp\left(ik\hat{r}\cdot\vec{x}'\right)\left\{\vec{\nabla}'\times\left[\vec{\nabla}'\times\left(\vec{D}-\varepsilon_0\vec{E}\right)\right]+i\frac{\varepsilon_0\omega}{c}\vec{\nabla}'\times\left(\vec{B}-\mu_0\vec{H}\right)\right\}$$

(We have made the usual approximations in evaluating $\frac{\exp\left(ik\left|\vec{x}-\vec{x}'\right|\right)}{\left|\vec{x}-\vec{x}'\right|}$). Now we do the "integration by parts" trick. The integral is of the form:

$$\begin{aligned}\int\exp\left(ik\hat{r}\cdot\vec{x}\right)\vec{\nabla}\times\left(\vec{\nabla}\times\vec{y}\right)dV &= \int\vec{\nabla}\times\left(e^{ik\hat{r}\cdot\vec{x}}\vec{\nabla}\times\vec{y}\right)dV-\int\vec{\nabla}e\times\left(\vec{\nabla}\times\vec{y}\right)dV\\ &= \int\hat{n}\times\left(e^{ik\hat{\mathbf{r}}\cdot\vec{x}}\vec{\nabla}\times\vec{y}\right)dS-ik\hat{r}\times\int e^{ik\hat{r}\cdot\vec{x}}\left(\vec{\nabla}\times\vec{y}\right)dV\\ &= 0-ik\hat{r}\times\int e^{ik\hat{r}\cdot\vec{x}}\left(\vec{\nabla}\times\vec{y}\right)dV\end{aligned}$$

Then we can do it again to get:

$$(ik)^2\,\hat{r}\times\left(\hat{r}\times\int e^{ik\hat{r}\cdot\vec{x}}\vec{y}dV\right)$$

Thus the expression for $\vec{A}$ is

$$\begin{aligned}\vec{A}_{sc} &= \frac{k^2}{4\pi}\int dV e^{ik\hat{\mathbf{r}}\cdot\vec{x}'}\left[-\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\left(\vec{D}-\varepsilon_0\vec{E}\right)\right)+\sqrt{\frac{\varepsilon_0}{\mu_0}}\hat{\mathbf{r}}\times\left(\vec{B}-\mu_0\vec{H}\right)\right]\\ &= \frac{k^2}{4\pi}\int dV e^{ik\hat{\mathbf{r}}\cdot\vec{x}'}\left[-\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\left(\vec{D}-\varepsilon_0\vec{E}\right)\right)+\sqrt{\frac{\varepsilon_0}{\mu_0}}\hat{\mathbf{r}}\times\left(\vec{B}-\mu_0\vec{H}\right)\right]\end{aligned}$$

(Jackson and I differ by a sign here.)

In this expression we can recognize the electric dipole moment (pg 4 of your multipole notes)

$$\vec{p}=\int e^{ik\hat{\mathbf{r}}\cdot\vec{x}'}\left(\vec{D}-\varepsilon_0\vec{E}\right)dV$$

and the magnetic dipole moment (pg 7)

$$\tilde{\mathbf{m}}=-\sqrt{\frac{\varepsilon_0}{\mu_0}}\int e^{ik\hat{\mathbf{r}}\cdot\vec{x}'}\left(\vec{B}-\mu_0\vec{H}\right)dV$$

The differential cross section for scattering into polarization $\hat{\boldsymbol{\varepsilon}}$ is:

$$\begin{aligned}\frac{d\sigma}{d} &= \frac{\left|\hat{\varepsilon}^*\cdot\vec{A}_{sc}\right|^2}{\left|\vec{D}^{(0)}\right|^2}\\ &= \frac{k^4}{\left|\vec{D}^{(0)}\right|^2}\left|\hat{\varepsilon}^*\cdot\vec{p}+(\hat{\mathbf{r}}\times\hat{\boldsymbol{\varepsilon}}^*)\cdot\vec{m}\right|\end{aligned}$$

Now to get an explicit solution we must approximate. We write

$$\vec{D}=\varepsilon_0\vec{E}+\delta\varepsilon\vec{E}$$

and similarly for $\vec{B}$. Then

$$\begin{aligned}\vec{D}-\varepsilon_0\vec{E} &= \delta\varepsilon\vec{E}=\frac{\delta\varepsilon}{\varepsilon_0}\vec{D}^{(0)}\\ &= \frac{\delta\varepsilon}{\varepsilon_0}D_0\hat{\mathbf{e}}_0e^{ik\hat{\mathbf{n}}_0\cdot\vec{x}}\end{aligned}$$

where we can use the unperturbed field on the right hand side since it is multiplied by the small quantity $\frac{\delta\varepsilon}{\varepsilon_0}$. Then the dipole moment is

$$\vec{p}=\int e^{ik\hat{\mathbf{r}}\cdot\vec{x}'}\frac{\delta\varepsilon}{\varepsilon_0}dV\,D_0\hat{e}_0e^{ik\hat{\mathbf{n}}_0\cdot\vec{x}}$$

Similarly

$$\begin{aligned}\vec{m} &= -\sqrt{\frac{\varepsilon_0}{\mu_0}}\int e^{ik\hat{\mathbf{r}}\cdot\vec{x}'}\frac{\delta\mu}{\mu_0}dVB_0\hat{\mathbf{n}}_0\times\hat{\mathbf{e}}_0 \\ &= -\int e^{ik\hat{\mathbf{r}}\cdot\vec{x}'}\frac{\delta\mu}{\mu_0}dVD_0\hat{\mathbf{n}}_0\times\hat{\mathbf{e}}_0\end{aligned}$$

since, from Faraday's law, $ik\hat{n}_0\times\vec{E}_0 = i\frac{\omega}{c}\vec{B}_0$, and so $\frac{\omega}{c}\sqrt{\mu_0\varepsilon_0}\hat{n}_0\times\vec{D}_0 = \varepsilon_0\frac{\omega}{c}\vec{B}_0$, $B_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}D_0$. Then the scattering cross section is:

$$\frac{d\sigma}{d} = k^4\left|\frac{1}{4\pi}\int dV'\exp\left(i\vec{q}\cdot\vec{x}'\right)\left[\frac{\delta\varepsilon}{\varepsilon_0}\hat{\mathbf{e}}_0\cdot\tilde{\boldsymbol{\varepsilon}}^* - \frac{\delta\mu}{\mu_0}\left(\hat{\mathbf{r}}\times\hat{\boldsymbol{\varepsilon}}^*\right)\cdot\left(\hat{\mathbf{n}}_0\times\hat{\mathbf{e}}_0\right)\right]\right|^2 \tag{9}$$

where

$$\vec{q} = k\left(\hat{\mathbf{n}}_0 - \hat{\mathbf{r}}\right)$$

If the wavelength $\lambda \gg$ the scale of the fluctuations $\delta\varepsilon$ and $\delta\mu$, then $e^{i\vec{q}\cdot\vec{x}} \approx 1$, and we have dipole radiation.

You will need equation (9) in problem 10.20.