Notes "orthogonal" January 2020

# 1 Expansion in orthogonal functions

To obtain a more useful form of the Green's function, we'll want to expand in orthogonal functions that are (relatively) easy to integrate. We begin with some basics and we'll see how to obtain solutions for the potential in boundary value problems with no charges inside the volume. Then we can move on to compute the Green's function.

## 1.1 Sturm-Liouville theory

See Lea Chapter 8 sections 1 and 2. Please review this material carefully.

## 1.2 General method for finding the potential

1. Choose coordinates so that the boundaries of the region correspond to constant-coordinate surfaces. Then write the defining equation

$$\nabla^2\Phi = 0$$

in terms of the chosen coordinates, called $u$, $v$, and $w$, separate variables, and solve to determine the eigenfunctions. You'll get equations something like:

$$\frac{D_u U}{U} = -(\alpha+\beta); \quad \frac{D_v V}{V} = \alpha; \quad \frac{D_w W}{W} = \beta$$

where $D_u$ is a 2nd-order differential operator of the form

$$f(u)\frac{d^2}{du^2} + g(u)\frac{d}{du} + h(u)$$

and similarly for $D_v$ and $D_w$.

2 Note which boundary has a *non-zero* value of potential. This boundary should be defined by the condition $u =$ constant for one of the coordinates, $u$. You should choose your separation constants so that the sets of eigenfunctions in the *other two* coordinates $v$ and $w$ are orthogonal functions.

3 Use the boundaries on which $\Phi = 0$ to determine the eigenfunctions $V$ and $W$, and the eigenvalues, (which I'll call $\alpha = \frac{D_v V}{V}$ and $\beta = \frac{D_w W}{W}$). If the boundaries are at a finite distance the eigenvalues will be countable and can be labelled with an integer, $\alpha_m$ for example . Otherwise they will form a continuous set.

4 The last eigenfunction is determined from the differential equation in the final coordinate, $u$:

$$\gamma = \frac{D_u U}{U} = -\alpha - \beta$$

At this point you should have a general solution of the form:

$$\Phi(u, v, w) = \sum_{m,n} A_{mn} U_{mn}(u : -\alpha_m - \beta_n) V_m(v : \alpha_m) W_n(w : \beta_n)$$

where the coefficients $A_{mn}$ are still undetermined.

5 Now use the final, non-zero, boundary condition, together with orthogonality of the eigenfunctions $V_m$ and $W_n$, to find the coefficients $A_{mn}$.

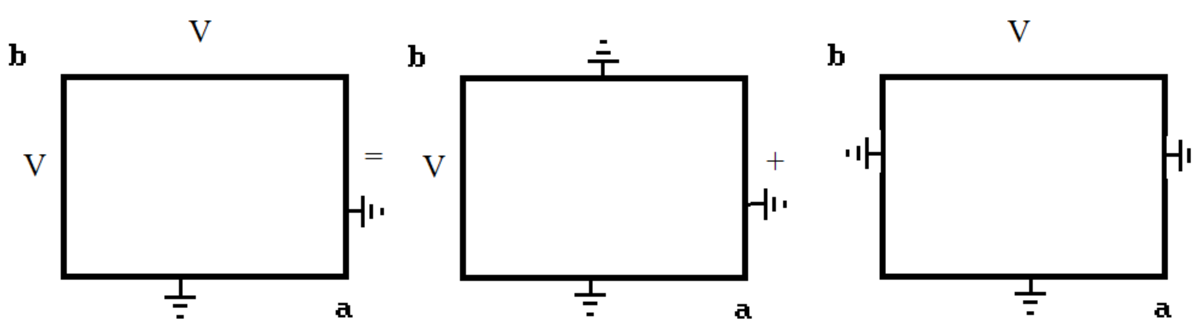
$$\begin{aligned} \Phi(u_0, v, w) &= \sum_{m,n} A_{mn} U_{mn}(u_0 : -\alpha_m - \beta_n) V_m(v : \alpha_m) W_n(w : \beta_n) \\ &= \text{known function of } v, w \end{aligned}$$

## 1.3 Rectangular coordinates

See Lea §8.2 and Griffiths Ch 3 §3, Jackson §2.9, or:
http://www.physics.sfsu.edu/~lea/courses/ugrad/360notes7.PDF

### 1.3.1 Rectangular 2-D problems with non-zero potential on more than one side.

Since the general method allows for non-zero potential on only one side, we have to solve these problems by superposition. Suppose we have a rectangular region measuring $a \times b$ with potential $V$ on the sides at $x = 0$ and $y = b$, the sides at $x = a$ and $y = 0$ being grounded. We solve two problems, each with three sides grounded, one having potential $V$ at $x = 0$ and one having potential $V$ at $y = b$. Then we add the results.




The solution is

$$\Phi = \Phi_1 + \Phi_2$$

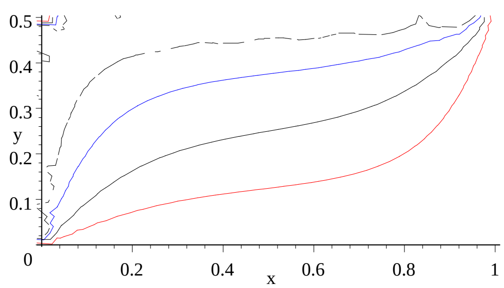
where (Lea Example 8.1)

$$\Phi_2 = \frac{4V}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin\left[\frac{(2m+1)\,\pi x}{a}\right] \frac{\sinh\frac{(2m+1)\pi y}{a}}{\sinh\frac{(2m+1)\pi b}{a}}$$

and we obtain $\Phi_2$ by interchanging $a$ and $b$, and noting that the potential is zero at $x = a$ and $y = 0$, so $x \to y$ and $y \to a - x$

$$\Phi_1 = \frac{4V}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[\frac{(2n+1)\,\pi y}{b}\right] \frac{\sinh\frac{(2n+1)\pi(a-x)}{b}}{\sinh\frac{(2n+1)\pi a}{b}}$$

With $a = 2b$ the potential looks like this. (The variables are $x/a$ and $y/a$, and the contours are at fixed values of $\Phi/V$.)




Blue 0.75, black, 0.5, red 0.25, dashed 0.9

### 1.3.2 Continuous set of eigenvalues

If one dimension of the rectangle becomes infinite, then the set of eigenvalues is no longer countable, but becomes a continuous set. Suppose we let $b \to \infty$ in the example above. With $\Phi(x, y) = X(x)\,Y(y)$, Laplace's equation takes the form

$$\frac{X''}{X} = k^2 = -\frac{Y''}{Y}$$

The solutions to the differential equation that satisfy the boundary conditions $\Phi = 0$ at $y = 0$ and at $x = a$ but $\Phi$ non-zero at $x = 0$ are of the form

$$\sin ky \sinh k(a - x)$$

But we no longer have an upper boundary in $y$ to determine the values of $k$. (Effectively, $\Phi_2$ in the previous problem has gone to zero because of the $\sinh kb$

in the denominator, leaving $\Phi_1$ with an undetermined $k$.) However, since the eigenfunction is an even function of $k$, we need only positive values of $k$. Thus the solution is

$$\Phi(x,y) = \int_0^\infty A(k) \sin ky \sinh [k(a-x)] \; dk \tag{1}$$

To find the function $A(k)$, we use the known potential $V(y)$ at $x = 0$ :

$$\Phi(0,y) = V(y) = \int_0^\infty A(k) \sin ky \sinh ka \; dk$$

We now make use of the orthogonality of the functions $\sin ky$ by multiplying both sides by $\sin k'y$ and integrating over $y$ :

$$\int_0^\infty V(y) \sin k'y \; dy = \int_0^\infty \int_0^\infty A(k) \sin ky \sinh ka \; dk \;\; \sin k'y \; dy$$

On the RHS, we have

$$\begin{aligned}
\int_0^\infty \sin ky \sin k'y \; dy &= \frac{1}{2}\int_0^\infty \left[\cos(k-k')y - \cos(k+k')y\right] \; dy \\
&= \frac{1}{4}\int_0^\infty \left\{e^{i(k-k')y} + e^{-i(k-k')y} - e^{i(k+k')y} - e^{-i(k+k')y}\right\} \; dy \\
&= \frac{1}{4}\int_{-\infty}^\infty \left\{\exp\left[i(k-k')y\right] - \exp\left[i(k+k')y\right]\right\} \; dy \\
&= \frac{\pi}{2}\left[\delta(k-k') - \delta(k+k')\right]
\end{aligned}$$

where we used Lea eqn 6.16. Thus

$$\int_0^\infty V(y) \sin k'y \; dy = \frac{\pi}{2}\int_0^\infty A(k) \sinh ka \left[\delta(k-k') - \delta(k+k')\right] \; dk$$

Since $k$ and $k'$ are both positive, only the first delta function contributes, and this integral reduces to

$$\int_0^\infty V(y) \sin k'y \; dy = \frac{\pi}{2} A(k') \sinh k'a$$

and thus, dropping the primes,

$$A(k) = \frac{2}{\pi}\int_0^\infty V(y) \frac{\sin ky}{\sinh ka} \; dy$$

For example, if $V(y) = V_0 e^{-y/h}$, then

$$\begin{aligned}
A(k) &= \frac{2}{\pi}\int_0^\infty V_0 e^{-y/h}\frac{\sin ky}{\sinh ka}\,dy \\
&= \frac{1}{\pi i}\frac{V_0}{\sinh ka}\left[\frac{e^{(ik-1/h)y}}{ik-\frac{1}{h}} - \frac{e^{(-ik-1/h)y}}{-ik-\frac{1}{h}}\right]_0^\infty \\
&= \frac{1}{\pi i}\frac{V_0}{\sinh ka}\left(\frac{-1}{ik-\frac{1}{h}} - \frac{1}{ik+\frac{1}{h}}\right) \\
&= \frac{1}{\pi i}\frac{V_0}{\sinh ka}\left(\frac{2ih^2k}{k^2h^2+1}\right)
\end{aligned}$$

and inserting this value into (1), we get

$$\Phi(x,y) = \frac{V_0}{\pi}\int_0^\infty \left(\frac{2hk}{k^2h^2+1}\right)\sin ky\frac{\sinh[k(a-x)]}{\sinh ka}\,hdk$$

*Analysis:* We can see right away that our result is dimensionally correct and real. Also, since $x \le a$, $k(a-x) \le ka$ and thus

$$\frac{\sinh[k(a-x)]}{\sinh ka} \le 1$$

for all $k$, so the integral converges. I couldn't do this integral analytically, so let's compute some values in the case $h = a$:

$\Phi\left(\frac{a}{2},\frac{a}{2}\right)/V_0 = \frac{2}{\pi}0.314\,45 = 0.200$

$\Phi\left(\frac{a}{2},a\right)/V_0 = \frac{2}{\pi}0.281\,18 = 0.179$

$\Phi\left(\frac{a}{2},2a\right)/V_0 = \frac{2}{\pi}0.119\,04 = 7.578\times 10^{-2}$

The potential decreases rapidly for $y > a$.

Check result at $x = 0$.

$$\begin{aligned}
\Phi(0,y) &= \frac{V_0}{\pi}\int_0^\infty \left(\frac{2h^2k}{k^2h^2+1}\right)\sin ky\frac{\sinh ka}{\sinh ka}\,dk \\
&= \frac{V_0}{\pi}\int_{-\infty}^\infty \frac{h^2k}{k^2h^2+1}\frac{e^{iky}-e^{-iky}}{2i}\,dk
\end{aligned}$$

We do the integral using countour integration. The integrand has poles at $k = \pm i/h$. For the first term we close up and for the second we close down. The integral along the big semicircle is zero by Jordan's lemma. Then the first term gives

$$2\pi i\frac{V_0}{2\pi i}\frac{h^2(i/h)e^{i(i/h)y}}{2h^2(i/h)} = \frac{V_0 e^{-h/y}}{2}$$

and for the second (remember we are going around the contour clockwise)

$$-2\pi i\frac{V_0}{2\pi i}\frac{h^2(-i/h)e^{-i(-i/h)y}}{2h^2(-i/h)} = -\frac{V_0 e^{-h/y}}{2}$$

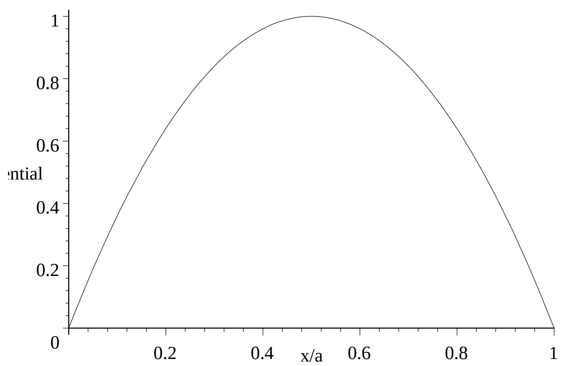
Thus

$$\Phi(0,y) = \frac{V_0 e^{-h/y}}{2} - \left(-\frac{V_0 e^{-h/y}}{2}\right) = V_0 e^{-h/y}$$

as required.

### 1.3.3 A 3-D problem

Find the potential inside a cubical box of side $a$ with grounded walls, except for the side at $z = a$ which has potential $V_0\left[1 - \left(\frac{2x}{a} - 1\right)^2\right]$. This potential increases from zero at the walls to $V_0$ at $x = a/2$,as shown in the diagram.




There is no charge inside the box, so the potential satisfies Laplace's equation:

$$\begin{aligned} \nabla^2\Phi &= 0 \\ \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} &= 0 \end{aligned}$$

Try separating variables:

$$\Phi = XYZ$$

Then

$$X''YZ + XY''Z + XYZ'' = 0$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \tag{2}$$

For equation (2) to hold for *all* values of $x, y,\ z$ we must have:

$$\frac{X''}{X} = k_1,\ \ \frac{Y''}{Y} = k_2,\ \ \frac{Z''}{Z} = k_3 \text{ and } k_1 + k_2 + k_3 = 0$$

The potential is zero at $x = 0$ and at $x = a$, so we need $k_1$ to be negative, $k_1 = -\alpha^2$, which gives the solutions $X = \sin\alpha x$ and $X = \cos\alpha x$, which have multiple zeros. We choose the sine function to make $X(0) = 0$, and then choose $\alpha = n\pi/a$ to make $X(a) = 0$. A similar reasoning gets $Y = \sin(m\pi y/a)$. Then

$$k_3 = \left(n^2 + m^2\right)\frac{\pi^2}{a^2}$$

and to make $Z(0) = 0$ the appropriate solution for $Z$ is the sinh.

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm}\sin\left(n\pi\frac{x}{a}\right)\sin\left(m\pi\frac{y}{a}\right)\sinh\left(\sqrt{n^2+m^2}\frac{\pi z}{a}\right)$$

Finally we evaluate the potential at $z = a$:

$$V_0\left[1 - \left(\frac{2x}{a} - 1\right)^2\right] = \sum_{n,m=1}^{\infty} A_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\sinh\left(\sqrt{n^2+m^2}\pi\right)$$

We make use of the orhogonality of the sines by multiplying both sides by $\sin(n'\pi x/a)\sin(m'\pi y/a)$ and integrating over $x$ and $y$. We may exchange the sum and the integrals on the RHS because the series is weakly convergent. The integrals are zero unless $n = n'$ and $m = m'$. Thus

$$\int_0^a\int_0^a V_0\left[1 - \left(\frac{2x}{a} - 1\right)^2\right]\sin\frac{n'\pi x}{a}\sin\frac{m'\pi y}{a}\,dxdy = A_{n'm'}\frac{a^2}{4}\sinh\sqrt{(n')^2 + (m')^2}\pi$$

Now we drop the primes on $n'$ and $m'$. The integral over $y$ is straighforward.

$$\int_0^a \sin\frac{m\pi y}{a}\,dy = \left.\frac{-\cos m\pi y/a}{m\pi/a}\right|_0^a = a\frac{1-(-1)^m}{m\pi}$$

The result is zero unless $m$ is odd. Let $u = 2x/a - 1$, to get (for $m$ odd),

$$\begin{aligned}
\frac{2a}{m\pi}\int_{-1}^{1} V_0\left(1-u^2\right)\sin\left[n\pi\left(\frac{u+1}{2}\right)\right]\frac{a}{2}du &= A_{nm}\frac{a^2}{4}\sinh\pi\sqrt{n^2+m^2} \\
\frac{4V_0}{m\pi}\int_{-1}^{1}\left(1-u^2\right)\left(\sin\frac{n\pi u}{2}\cos\frac{n\pi}{2} + \cos\frac{n\pi u}{2}\sin\frac{n\pi}{2}\right)du &= A_{nm}\sinh\pi\sqrt{n^2+m^2} \quad (3)
\end{aligned}$$

When we integrate over $u$ the $\sin n\pi u/2$ term gives zero because we have an odd integrand over an even range. Then $\sin n\pi/2$ is zero unless $n$ is odd. The remaining integrand is even, so we can use half the range.

$$\int_{-1}^{1}\left(1-u^2\right)\cos\frac{n\pi u}{2}du = 2\int_0^1\left(1-u^2\right)\cos\frac{n\pi u}{2}du$$

We do integation by parts:

$$\int_0^1\left(1-u^2\right)\cos\frac{n\pi u}{2}du = \left.\frac{2\left(1-u^2\right)}{n\pi}\sin\frac{n\pi u}{2}\right|_0^{+1} + \frac{4}{n\pi}\int_0^1 u\sin\frac{n\pi u}{2}du$$

The integrated term is zero. We do another integration by parts to get

$$\begin{aligned}
\int_0^1 \left(1-u^2\right)\cos\frac{n\pi u}{2}du &= \frac{4}{n\pi}\left(u\frac{-2}{n\pi}\cos\frac{n\pi u}{2}\bigg|_0^{+1}+\frac{2}{n\pi}\int_0^1\cos\frac{n\pi u}{2}du\right) \\
&= -\frac{8}{(n\pi)^2}\left(\cos\frac{n\pi}{2}-\left(\frac{2}{n\pi}\right)\sin\frac{n\pi u}{2}\bigg|_0^{+1}\right) \\
&= \frac{16}{(n\pi)^3}\sin\frac{n\pi}{2}
\end{aligned}$$

where the first term is zero for $n$ odd. So the left hand side of (3) is

$$\frac{4V_0}{m\pi}\frac{32}{(n\pi)^3}\sin^2 n\frac{\pi}{2} = V_0\frac{128}{mn^3\pi^4} \text{ for } m \text{ and } n \text{ odd, and zero otherwise}$$

Thus for $m$ odd and $n$ odd

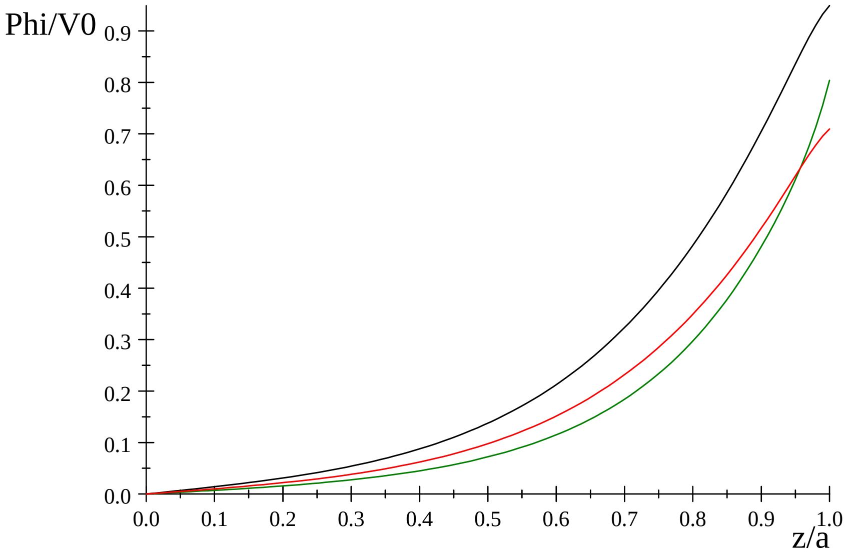
$$A_{nm} = V_0\frac{128}{mn^3\pi^4}\frac{1}{\sinh\sqrt{n^2+m^2}\pi}$$

and finally we have the potential

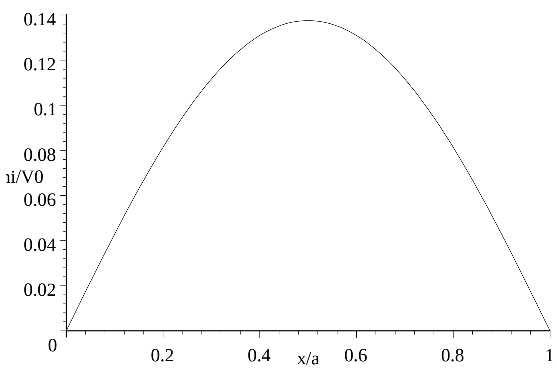
$$\Phi(x,y,z) = \frac{128}{\pi^4}V_0\sum_{m=1,\text{odd}}^{\infty}\sum_{n=1,\text{odd}}^{\infty}\frac{1}{mn^3}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\frac{\sinh\sqrt{n^2+m^2}\frac{\pi z}{a}}{\sinh\sqrt{n^2+m^2}\pi}$$

*Analysis:* The result is dimensionally correct, and converges nicely.

The plot shows the series with values of $m$ up to $m = 11$ and $n$ up to $n = 5$. The black line is the potential as a function of $z$ for $x = y = a/2$,the green line for $x = y = a/4$,and the red line for $x = a/2$, $y = a/4$ . The potential increases faster near the center of the box, as expected, since the potential on the top side is maximum at $x = a/2$. .




At $z = a/2, y = a/2$ the potential versus $x/a$ looks like this:




The potential is symmetric about $x = a/2$, as expected.

# 2 Complex potential

Recall that both the real and imaginary parts of an analytic function satisfy Laplace's equation in two dimensions (Lea Chapter 2 section 4) . We may use this fact to solve 2-d potential problems.

#### 2.0.4 Potential in a wedge.

Suppose the region of interest is defined by the angular wedge $0 \leq \theta \leq \alpha$. Further suppose the potential is zero on the conducting boundaries at $\theta = 0$ and $\theta = \alpha$. Then the analytic function

$$f(z) = z^{\pi/\alpha} = r^{\pi/\alpha} e^{i\theta\pi/\alpha} = r^{\pi/\alpha}\left(\cos\frac{\pi}{\alpha}\theta + i\sin\frac{\pi}{\alpha}\theta\right)$$

has imaginary part

$$v(r,\theta) = r^{\pi/\alpha}\sin\frac{\pi}{\alpha}\theta$$

that satisfies the boundary conditions. If $\alpha = \pi/m$ for some integer $m$, then $f(z)$ is analytic everywhere. If $\alpha$ is an arbitrary real number, then $f(z)$ may have a branch point at the origin, but we may choose the branch cut so that $f(z)$ is still analytic in our region everywhere except AT the origin. Thus $v(r,\theta)$ also satisfies Laplace's equation in the volume, and so it is a solution of the correct form. In fact the function $f(z) = z^{n\pi/\alpha}$ for any integer $n$ has the same nice properties. Thus the potential in a wedge-shaped region with opening angle $\alpha$ and conducting boundaries at potential $V_0$ is described by the complex potential

$$\Phi(z) = A + iV_0 + \Sigma_n a_n z^{n\pi/\alpha}$$

The imaginary part is

$$V_0 + \sum_{n=-\infty}^{+\infty} a_n r^{n\pi/\alpha}\sin\frac{n\pi}{\alpha}\theta$$

which is the potential in the region. The coefficients $a_n$ must be chosen to satisfy any remaining boundary conditions in $r$. Compare with Jackson's eqn. 2.72 which he derives using separation of variables in plane polar coordinates. (You should read this derivation carefully.)

If the origin is included within our region, the sum is over positive $n$ only, so that the potential remains finite. Then the potential near the origin (small $r$) is dominated by the first ($n = 1$) term, and the field near the origin has components

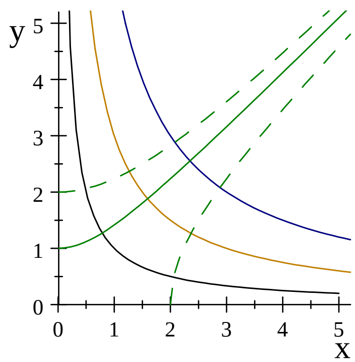
$$\vec{E} = -\vec{\nabla}\Phi = -a_1\frac{\pi}{\alpha}r^{\pi/\alpha-1}\left[\sin\left(\frac{\pi}{\alpha}\theta\right)\ \hat{r} + \cos\left(\frac{\pi}{\alpha}\theta\right)\ \hat{\theta}\right]$$

Thus $\left|\vec{E}\right| \to 0$ as $r \to 0$ if $\pi/\alpha > 1$ but $\left|\vec{E}\right| \to \infty$ as $r \to 0$ if $\pi/\alpha < 1$. The field is small in a hole ($\alpha < \pi$) but very large near a spike ($\alpha > \pi$).

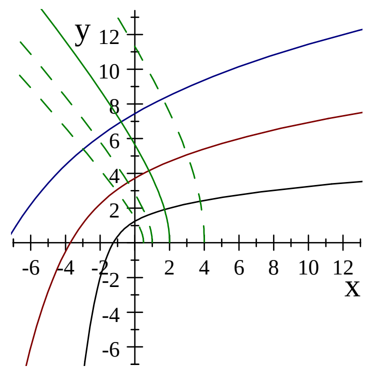
The diagrams show the cases of $\alpha = \pi/2$ and $3\pi/2$. For $\alpha = \pi/2$, the equipotential surfaces are given by

$$r^2 \sin 2\theta = \text{constant} = 2xy$$

Thus the equipotentials are hyperbolae in this case.




equipotentials and field lines for $\alpha = \pi/2$




$\alpha = 3\pi/2$

We may use conformal mapping to find the potential in a 2-d boundary value problem with more complex boundaries. See Lea Ch 2 section 2.8. We choose the mapping to simplify the boundary shape, solve the simpler problem, and then map back.