Multipole fields

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1 Introduction

The big idea is to time- transform all physical quantities, such as the current vector and the fields. We have

$$J^{\mu}\left(\tilde{\mathbf{x}},t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}J^{\mu}\left(\tilde{\mathbf{x}},\omega\right)\exp\left(-i\omega t\right)dt$$

The wave equation for $\mathbf{A}$ is:

$$\Box^{2}\mathbf{A}=\frac{4\pi}{c}\mathbf{J}$$

and we already have the solution

$$\mathbf{A}=\frac{4\pi}{c}\int\mathbf{J}\left(\mathbf{x}'\right)\frac{\delta\left(\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|-c\left(t-t'\right)\right)}{4\pi\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}d^{4}\mathbf{x}'$$

which we may write in terms of the time transform of $\mathbf{J}$

$$A^{\mu}\left(\mathbf{x}\right)=\frac{1}{c}\int\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}J^{\mu}\left(\tilde{\mathbf{x}}',\omega\right)e^{-i\omega t'}d\omega\frac{\delta\left(\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|-c\left(t-t'\right)\right)}{\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}d^{4}\mathbf{x}'$$

Now we do the integration over t' :

$$\begin{aligned}A^{\mu}\left(\mathbf{x}\right) &= \frac{1}{c}\int\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}J^{\mu}\left(\tilde{\mathbf{x}}',\omega\right)e^{-i\omega t'}d\omega\frac{\delta\left(ct'-c(t-\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|/c)\right)}{\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}dct'd^{3}\tilde{\mathbf{x}}' \\ &= \frac{1}{\sqrt{2\pi}}\int\int_{-\infty}^{+\infty}\frac{J^{\mu}\left(\tilde{\mathbf{x}}',\omega\right)\exp\left(-i\omega(t-\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|/c)\right)}{c\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}d\omega d^{3}\tilde{\mathbf{x}}' \qquad (1)\end{aligned}$$

The 0th component is:

$$\Phi\left(\mathbf{x}\right)=\frac{1}{\sqrt{2\pi}}\int\int_{-\infty}^{+\infty}d\omega e^{-i\omega t}\frac{\rho\left(\tilde{\mathbf{x}}',\omega\right)\exp\left(i\omega\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|/c\right)}{\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}d^{3}\tilde{\mathbf{x}}' \qquad (2)$$

while the 1,2,3 components are:

$$\tilde{\mathbf{A}}=\frac{1}{\sqrt{2\pi}}\int\int_{-\infty}^{+\infty}d\omega e^{-i\omega t}\frac{\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}',\omega\right)\exp\left(i\omega\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|/c\right)}{c\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}d^{3}\tilde{\mathbf{x}}' \qquad (3)$$

Transforming the relations between the fields and the potentials, we get:

$$\tilde{\mathbf{B}}\left(\tilde{\mathbf{x}},\omega\right)=\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{A}}\left(\tilde{\mathbf{x}},\omega\right)$$

and

$$\frac{1}{c}\frac{\partial \tilde{\mathbf{E}}}{\partial t} = \tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{B}} \to -\frac{i\omega}{c}\tilde{\mathbf{E}}\left(\tilde{\mathbf{x}},\omega\right) = \tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{B}}\left(\tilde{\mathbf{x}},\omega\right)$$

and thus

$$\tilde{\mathbf{E}}\left(\tilde{\mathbf{x}},\omega\right) = \frac{ic}{\omega}\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{B}}\left(\tilde{\mathbf{x}},\omega\right)$$

We have 3 relevant length scales: d, the dimension of the source, λ, the wavelength, and r, the distance from source to observer. The ordering of these lengths determines how we proceed.

- $d \ll r \ll \lambda$. This is the near, or static region. With $r \ll \lambda$, the exponential in equations(3) and (2) is $\exp(2\pi i r/\lambda) \approx 1$ and we get the static results from Chapter 3 for the time transform $\tilde{\mathbf{A}}\left(\tilde{\mathbf{x}},\omega\right)$. Thus we get the static fields, but oscillating in time.
- $d \ll r \sim \lambda$. The induction zone. This is tricky.
- $d \ll \lambda \ll r$. The radiation zone. In this zone the source appears almost point-like. We may expand the quantity $\left|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'\right|$:

$$\begin{aligned} \left|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'\right|^2 &= r^2 + x'^2 - 2\tilde{\mathbf{r}} \cdot \tilde{\mathbf{x}}' \\ \left|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'\right| &= r\left(1 - \frac{\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}'}{r}\right) = r - \hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}' \end{aligned}$$

and similarly

$$\frac{1}{\left|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'\right|} = \frac{1}{r} + \hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}' + \cdots \approx \frac{1}{r}$$

Remember that we need more accuracy in the exponential than in the quantity outside the exponential. From here on we shall assume we are in the radiation zone.

2 The dipole fields

We put the approximations for $\left|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'\right|$ into the expression for $\tilde{\mathbf{A}}$ (equation 3):

$$\tilde{\mathbf{A}} = \frac{1}{\sqrt{2\pi}}\int\int_{-\infty}^{+\infty} d\omega e^{-i\omega t}\frac{\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}',\omega\right)\exp\left(ikr - ik\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}'\right)}{cr}d^3\tilde{\mathbf{x}}'$$

Thus the time transform of $\tilde{\mathbf{A}}$ is:

$$\tilde{\mathbf{A}}\left(\tilde{\mathbf{x}},\omega\right) = \frac{e^{ikr}}{cr}\int \tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}',\omega\right)\exp\left(-ik\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}'\right)d^3\tilde{\mathbf{x}}'$$

where $k = \omega/c$. Now we expand the exponential in the integrand:

$$\begin{aligned} \exp\left(-ik\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}'\right) &= 1 - ik\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}' + \frac{1}{2}\left(-ik\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}'\right)^2 + \cdots \\ &= 1 - ik\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}' - \frac{k^2}{2}\left(\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}'\right)^2 + \cdots \end{aligned}$$

The first term in $\tilde{\mathbf{A}}$ is:

$$\tilde{\mathbf{A}}_d(\tilde{\mathbf{x}},\omega) = \frac{e^{ikr}}{cr}\int \tilde{\mathbf{J}}(\tilde{\mathbf{x}}',\omega)\, d^3\tilde{\mathbf{x}}'$$

We can simplify this expression by using the equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}} = 0$$

Taking the time transform, we have:

$$-i\omega\rho + \tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}(\tilde{\mathbf{x}},\omega) = 0$$

Now

$$\begin{aligned}
\tilde{\boldsymbol{\nabla}}\left(\tilde{\mathbf{x}}\cdot\tilde{\mathbf{J}}\right) &= \left(\tilde{\mathbf{x}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{\mathbf{J}} + \left(\tilde{\mathbf{J}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{\mathbf{x}} + \tilde{\mathbf{x}}\times\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right) + \tilde{\mathbf{J}}\times\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{x}}\right) \\
&= \left(\tilde{\mathbf{x}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{\mathbf{J}} + \tilde{\mathbf{J}} + \tilde{\mathbf{x}}\times\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right) \\
&= x_m\partial_m J_i + J_i + \varepsilon_{ijk}x_j\varepsilon_{klm}\partial_l J_m \\
&= x_m\partial_m J_i + J_i + \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)x_j\partial_l J_m \\
&= x_m\partial_m J_i + J_i + x_j\partial_i J_j - x_l\partial_l J_i \\
&= J_i + x_j\partial_i J_j + x_i\partial_m J_m - x_i\partial_m J_m \\
&= \tilde{\mathbf{J}} + \tilde{\mathbf{x}}\left(\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}\right) + x_j\partial_i J_j - x_i\partial_m J_m \\
&= \tilde{\mathbf{J}} + \tilde{\mathbf{x}}\left(\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}\right) + \left(\tilde{\mathbf{x}}\times\tilde{\boldsymbol{\nabla}}\right)\times\tilde{\mathbf{J}} \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\text{all space}}\tilde{\boldsymbol{\nabla}}\left(\tilde{\mathbf{x}}\cdot\tilde{\mathbf{J}}\right)dV &= \int_{s_\infty}\left(\tilde{\mathbf{x}}\cdot\tilde{\mathbf{J}}\right)\hat{\mathbf{n}}\,dA = 0 \\
&= \int_{\text{all space}}\left(\tilde{\mathbf{J}} + \tilde{\mathbf{x}}\left(\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}\right) + \left(\tilde{\mathbf{x}}\times\tilde{\boldsymbol{\nabla}}\right)\times\tilde{\mathbf{J}}\right)dV
\end{aligned}$$

since $\mathbf{J}$ is zero outside the source. Thus:

$$\begin{aligned}
\int\tilde{\mathbf{J}}(\tilde{\mathbf{x}}',\omega)\,d^3\tilde{\mathbf{x}}' &= -\int\tilde{\mathbf{x}}'\left(\tilde{\boldsymbol{\nabla}}'\cdot\tilde{\mathbf{J}}\right)d^3\tilde{\mathbf{x}}' \\
&= -i\int\tilde{\mathbf{x}}'\omega\rho(\tilde{\mathbf{x}}',\omega)\,d^3\tilde{\mathbf{x}}' = -i\omega\tilde{\mathbf{p}}(\omega)
\end{aligned}$$

where $\tilde{\mathbf{p}}$ is the dipole moment of the source.

Then:

$$\tilde{\mathbf{A}}_d(\tilde{\mathbf{x}},\omega) = -i\omega\frac{e^{ikr}}{cr}\tilde{\mathbf{p}}(\omega) = -ik\frac{e^{ikr}}{r}\tilde{\mathbf{p}}(\omega)$$

$$\tilde{\mathbf{B}} = \tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{A}} = -\frac{ik}{r}\left(ik - \frac{1}{r}\right)e^{ikr}\hat{\mathbf{r}}\times\tilde{\mathbf{p}}$$

and in the radiation zone $k \gg 1/r$, so:

$$\tilde{\mathbf{B}} = k^2\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\tilde{\mathbf{p}}$$

and then

$$\begin{aligned}
\tilde{\mathbf{E}}\left(\tilde{\mathbf{x}},\omega\right) &= \frac{i}{k}\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{B}}\left(\tilde{\mathbf{x}},\omega\right) = \frac{i}{k}\tilde{\boldsymbol{\nabla}}\times\left(k^2\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\tilde{\mathbf{p}}\right) \\
&= ik\left(ik\frac{e^{ikr}}{r}\right)\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{p}}\right) \\
&= -k^2\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{p}}\right)
\end{aligned}$$

3 Power radiated

The power radiated per unit solid angle is given by the Poynting vector:

$$\frac{dP}{d\Omega} = r^2\left|\tilde{\mathbf{S}}\right| = r^2\frac{c}{4\pi}\left|\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right|$$

The the total energy radiated is

$$\frac{dW}{d\Omega} = \frac{c}{4\pi}\int_{-\infty}^{+\infty} r^2\left|\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right| dt$$

and using Parseval's theorem we may convert to an integral of the transforms over frequency:

$$\frac{dW}{d\Omega} = \frac{c}{4\pi}\int_{-\infty}^{+\infty} r^2\left|\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\right| d\omega$$

where

$$\begin{aligned}
r^2\left|\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\right| &= r^2\frac{c}{4\pi}\left|\left(-k^2\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{p}}\right)\right)\times\left(k^2\frac{e^{-ikr}}{r}\hat{\mathbf{r}}\times\tilde{\mathbf{p}}^*\right)\right| \\
&= \frac{c}{4\pi}k^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}\right|^2
\end{aligned}$$

and so

$$\frac{d^2W}{d\Omega d\omega} = \frac{c}{4\pi}k^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}\right|^2 \tag{5}$$

is the energy radiated per unit solid angle and per unit frequency.

4 Periodic source

If the source is periodic, then the current must be expanded in a Fourier series rather than a Fourier transform. We have

$$\mathbf{J}\left(\tilde{\mathbf{x}},t\right) = \sum_{n=-\infty}^{+\infty}\mathbf{J}_n\left(\tilde{\mathbf{x}}\right)\exp\left(in\omega_0 t\right)$$

where ω_0 is the fundamental frequency = $2\pi/T$ and T is the period of the source. Then the expression for $\tilde{\mathbf{A}}$ becomes:

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{1}{c}\int \sum_{n=-\infty}^{+\infty} \mathbf{J}_n(\tilde{\mathbf{x}}')\exp(in\omega_0 t')\frac{\delta\left(\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|-c(t-t')\right)}{\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}d^4\mathbf{x}' \\
&= \sum_{n=-\infty}^{+\infty}\int \frac{\mathbf{J}_n(\tilde{\mathbf{x}})\exp(in\omega_0 t)}{c\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}\exp\left(-in\omega_0\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|/c\right)d^3\tilde{\mathbf{x}}'
\end{aligned}$$

which is a Fourier series for $\mathbf{A}$ with coefficients:

$$\mathbf{A}_n = \sum_{n=-\infty}^{+\infty}\int \frac{\mathbf{J}_n(\tilde{\mathbf{x}})}{c\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}\exp\left(-in\omega_0\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|/c\right)d^3\tilde{\mathbf{x}}'$$

Following the analysis above, we find the dipole term to be:

$$\tilde{\mathbf{A}}_{n,d}(\tilde{\mathbf{x}}) = -in\omega_0\frac{e^{-ik_n r}}{cr}\tilde{\mathbf{p}}_n = -ink_0\frac{e^{-ik_n r}}{r}\tilde{\mathbf{p}}_n$$

where $k_n = n\omega_0/c$ and

$$\tilde{\mathbf{p}}_n = \int \tilde{\mathbf{x}}'\rho_n(\tilde{\mathbf{x}}')\,d^3\tilde{\mathbf{x}}'$$

with ρ_n being the nth coefficient in the Fourier series for ρ. Then we find the power radiated is:

$$\begin{aligned}
\frac{dP}{d\Omega} &= r^2\frac{c}{4\pi}\left|\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right| \\
&= r^2\frac{c}{4\pi}\left|\left(\sum_{n=-\infty}^{+\infty} -k_n^2\frac{e^{-ik_n r}}{r}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_n)\,e^{in\omega_o t}\right)\times\left(\sum_{m=-\infty}^{+\infty} k_m^2\frac{e-^{ik_m r}}{r}\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_m e^{im\omega_0 t}\right)\right| \\
&= \frac{c}{4\pi}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} k_n^2 k_m^2 e^{-ik_n r}e^{-ik_m r}\left|-(\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_n))\times(\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_m)\right|\exp(i\omega_0 t(n+m))
\end{aligned}$$

Now we time average by integrating over one period T and dividing by T. Then only those terms with $m=-n$ survive the integration. We have:

$$<\frac{dP}{d\Omega}>=\frac{c}{4\pi}\sum_{n=-\infty}^{+\infty}k_n^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_n\right|^2 \tag{6}$$

This equation is very similar to equation (5), but the meaning is somewhat different. Equation (5) is the energy radiated per unit frequency – the power spectrum. On the other hand equation (6) gives the time averaged power in the nth harmonic for a periodic source. Note however, that the real physical frequency ω_n is described by both the negative and the positive frequency $\pm\omega_n$,and for a real dipole $\tilde{\mathbf{p}}_n^* = \tilde{\mathbf{p}}_{-n}$,so

$$<\frac{dP_n}{d\Omega}>=\frac{c}{2\pi}k_n^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_n\right|^2 \tag{7}$$

In this expression $\bar{\mathbf{p}}_n$ is the coefficient of the *exponential* Fourier series. Careful use of these expressions obviates any of the factor-of-2 problems which otherwise arise.

Example: Suppose we have a pure frequency dipole: $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}_0\cos(\omega_0 t)\;=$

$\frac{\tilde{\mathbf{p}}_0}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)$. Thus $\tilde{\mathbf{p}}_1 = \tilde{\mathbf{p}}_{-1} = \tilde{\mathbf{p}}_0/2$. Then

$$\tilde{\mathbf{A}} = -ink_0\frac{1}{2r}\tilde{\mathbf{p}}_0\left(e^{ik_0 r}e^{i\omega_0 t}-e^{-ik_0 r}e^{-i\omega_0 t}\right)$$

and

$$\begin{aligned}\frac{dP}{d\Omega} &= r^2\frac{c}{4\pi}\left|\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right| \\ &= r^2\frac{c}{4\pi}\left|\left(-k_0^2\frac{1}{2r}\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_0\right)\left(e^{ik_0 r}e^{i\omega_o t}-e^{-ik_0 r-i\omega_0 t}\right)\right)\times\left(k_0^2\frac{1}{2r}\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_0\left(e^{ik_0 r}e^{i\omega_o t}-e^{-ik_0 r-i\omega_0 t}\right)\right)\right| \\ &= \frac{k_0^4}{4}\frac{c}{4\pi}\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_0\right|^2\left|2-e^{2(ik_0 r)}e^{i2\omega_o t}-e^{-2(ik_0 r)}e^{-i2\omega_o t}\right|\end{aligned}$$

Now we time average to get:

$$<\frac{dP}{d\Omega}>=2\frac{c}{16\pi}k_0^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_0\right|^2=\frac{c}{8\pi}k_0^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_0\right|^2$$

which agrees with Jackson's equation 9.22.

Alternatively, we can use equation (7) directly. We get:

$$<\frac{dP_n}{d\Omega}>=\frac{c}{2\pi}k_n^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_n\right|^2=\frac{c}{2\pi}\left(\frac{\omega_0}{c}\right)^4\left|\hat{\mathbf{r}}\times\frac{\tilde{\mathbf{p}}_0}{2}\right|^2=\frac{c}{8\pi}\left(\frac{\omega_0}{c}\right)^4\left|\hat{\mathbf{r}}\times\tilde{\mathbf{p}}_0\right|^2$$

as expected.

5 Quadrupole fields

The second term in the expansion of $\tilde{\mathbf{A}}$ is:

$$\tilde{\mathbf{A}}_2=\frac{e^{ikr}}{cr}\int\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}',\omega\right)\left(-ik\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}'\right)d^3\tilde{\mathbf{x}}'$$

To evaluate this term, first we work on the inetgrand. Note that

$$\left(\tilde{\mathbf{x}}'\times\tilde{\mathbf{J}}\right)\times\hat{\mathbf{r}}=\tilde{\mathbf{J}}\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}'\right)-\tilde{\mathbf{x}}'\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{J}}\right)$$

Thus we can write

$$\frac{1}{c}\tilde{\mathbf{J}}\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}'\right)=\frac{1}{2c}\left\{\tilde{\mathbf{J}}\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}'\right)+\tilde{\mathbf{x}}'\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{J}}\right)+\left(\tilde{\mathbf{x}}'\times\tilde{\mathbf{J}}\right)\times\hat{\mathbf{r}}\right\}$$

The antisymmetric part of this expression is

$$\frac{1}{2c}\left(\tilde{\mathbf{x}}'\times\tilde{\mathbf{J}}\right)\times\hat{\mathbf{r}}=\tilde{\mathbf{M}}\times\hat{\mathbf{r}}$$

where $\tilde{\mathbf{M}}$ is the magnetization (cf Jackson equation 5.53). Thus:

$$\begin{aligned}\tilde{\mathbf{A}}_2 &= -ik\frac{e^{ikr}}{r}\left(\int\frac{1}{2c}\left\{\tilde{\mathbf{J}}\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}'\right)+\tilde{\mathbf{x}}'\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{J}}\right)\right\}d^3\tilde{\mathbf{x}}'+\tilde{\mathbf{m}}\times\hat{\mathbf{r}}\right) \\ &= \tilde{\mathbf{A}}_q+\tilde{\mathbf{A}}_{md}\end{aligned}$$

where

$$\tilde{\mathbf{A}}_{md}=-ik\frac{e^{ikr}}{r}\tilde{\mathbf{m}}\times\hat{\mathbf{r}}$$

is the magnetic dipole term and

$$\tilde{\mathbf{A}}_q = -ik\frac{e^{ikr}}{r}\int \frac{1}{2c}\left\{\tilde{\mathbf{J}}\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}'\right)+\tilde{\mathbf{x}}'\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{J}}\right)\right\}d^3\tilde{\mathbf{x}}'$$

is the quadrupole term.

Analysis of the magnetic dipole term proceeds exactly as for the electric dipole term, with $\tilde{\mathbf{p}}$ replaced by $\tilde{\mathbf{m}}\times\hat{\mathbf{r}}$. The extra cross product with $\hat{\mathbf{r}}$ changes the directions of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, i.e. the polarization of these fields differs from that of the electric dipole terms.

$$\tilde{\mathbf{B}}_{md} = k^2\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\left(\tilde{\mathbf{m}}\times\hat{\mathbf{r}}\right) = k^2\frac{e^{ikr}}{r}\left(\tilde{\mathbf{m}}-\hat{\mathbf{r}}\left(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}}\right)\right)$$

while

$$\tilde{\mathbf{E}}_{md} = -k^2\frac{e^{ikr}}{r}\left(\hat{\mathbf{r}}\times\tilde{\mathbf{m}}\right)$$

and the power is given by equation (4) or (6) with $\tilde{\mathbf{p}}$ replaced by $\tilde{\mathbf{m}}$.

Now we proceed with the quadrupole term. We are supposed to do an integration by parts, so let's run through the usual steps. First note that:

$$\begin{aligned}\partial_i\left(x_kJ_i\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)\right) &= \delta_{ik}J_i\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)+x_k\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}+x_kJ_i\left(\hat{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{\mathbf{x}}\\ &= J_k\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)+x_k\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}+x_k\tilde{\mathbf{J}}\cdot\hat{\mathbf{r}}\end{aligned}$$

So

$$\begin{aligned}\int\partial_i\left[x_kJ_i\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)\right]dV &= \int_S\left(x_kJ_i\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)\right)dA_i = 0\\ &= \int\left(J_k\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)+x_k\left(\hat{\mathbf{r}}\cdot\tilde{\mathbf{x}}\right)\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}+x_k\tilde{\mathbf{J}}\cdot\hat{\mathbf{r}}\right)dV\end{aligned}$$

Then:

$$\begin{aligned}\vec{A}_q &= -ik\frac{e^{ikr}}{r}\int\frac{1}{2c}\left\{\vec{J}\left(\hat{\mathbf{r}}\cdot\vec{x}'\right)+\vec{x}'\left(\hat{\mathbf{r}}\cdot\vec{J}\right)\right\}d^3\vec{x}'\\ &= ik\frac{e^{ikr}}{2cr}\int\vec{x}\left(\hat{\mathbf{r}}\cdot\vec{x}\right)\left(\vec{\nabla}\cdot\vec{J}\right)dV\\ &= ik\frac{e^{ikr}}{2cr}\int\vec{x}\left(\hat{\mathbf{r}}\cdot\vec{x}\right)i\omega\rho dV\\ &= -k^2\frac{e^{ikr}}{2r}\int\vec{x}\left(\hat{\mathbf{r}}\cdot\vec{x}\right)\rho dV\end{aligned}$$

Then

$$\vec{B} = \nabla\times\vec{A} = -\frac{ik^3}{2r}e^{ikr}\int\left(\hat{\mathbf{r}}\times\vec{x}\right)\left(\hat{\mathbf{r}}\cdot\vec{x}\right)\rho dV$$

Now define the vector $\vec{Q}\left(\hat{\mathbf{r}}\right)$ by

$$Q_i = \sum_j Q_{ij}\hat{r}_j \tag{8}$$

where Q_{ij} is the quadrupole tensor

$$Q_{ij} = \int\left(3x_ix_j-\left|\tilde{\mathbf{x}}\right|^2\delta_{ij}\right)\rho\left(\tilde{\mathbf{x}}\right)d^3\tilde{\mathbf{x}}$$

Then

$$
\begin{aligned}
Q_i &= \sum_j \int \left(3x_i x_j \hat{r}_j - |\tilde{\mathbf{x}}|^2 \hat{r}_i\right) \rho\left(\tilde{\mathbf{x}}\right) d^3\tilde{\mathbf{x}} \\
\vec{Q} &= \int 3\tilde{\mathbf{x}}\left(\hat{\mathbf{r}} \cdot \tilde{\mathbf{x}}\right) \rho dV - \hat{\mathbf{r}} \int |\tilde{\mathbf{x}}|^2 \rho dV
\end{aligned}
$$

and thus

$$
\vec{B} = -\frac{ik^3}{2r} e^{ikr} \frac{\hat{\mathbf{r}} \times \vec{Q}}{3}
$$

and then

$$
\begin{aligned}
\vec{E} &= \frac{i}{k}\left(\frac{k^4}{2r} e^{ikr} \frac{\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \vec{Q}\right)}{3}\right) \\
&= \frac{ik^3}{6r} e^{ikr} \hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \vec{Q}\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{d^2W}{d\Omega d\omega} &= \frac{c}{4\pi} \frac{k^6}{36} \left|\hat{\mathbf{r}} \times \vec{Q}\right|^2 \\
&= \frac{c}{144\pi} k^6 \left|\hat{\mathbf{r}} \times \vec{Q}\right|^2
\end{aligned}
$$

or, for a periodic source, the time averaged power is

$$
< \frac{dP_n}{d\Omega} >= \frac{c}{72\pi} k^6 \left|\hat{\mathbf{r}} \times \vec{Q}_n\right|^2
$$

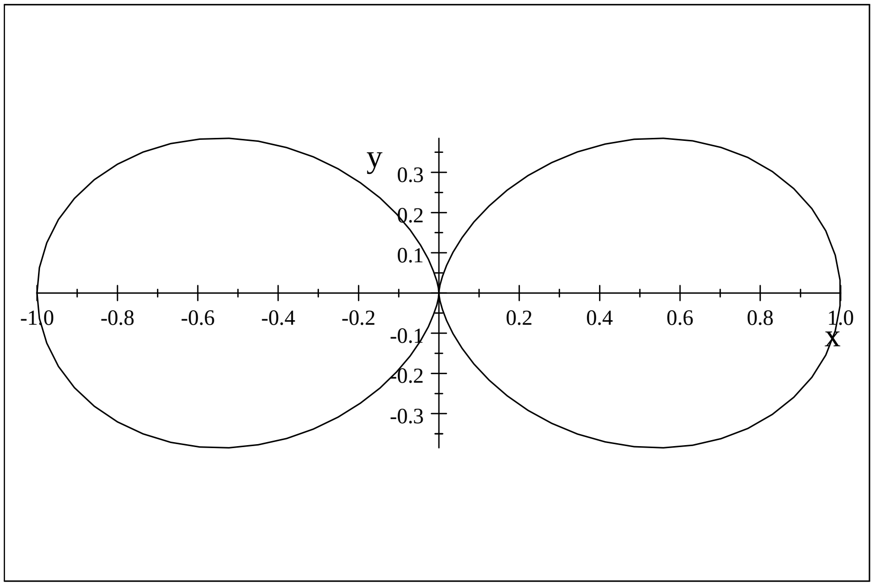
where again Q_n is the coefficient in the exponential Fourier series.

6 Angular distribution

For the dipole:

$$
\frac{dP}{d\Omega} \propto |\hat{\mathbf{r}} \times \vec{p}|^2 = p^2 \sin^2 \theta
$$

where θ is the angle between $\vec{p}$ and the direction of propagation $\hat{\mathbf{r}}$.



Dipole radiation pattern

The total power radiated is:

$$\begin{aligned} P_n &= \int \frac{dP_n}{d\Omega} d\Omega = \frac{c}{2\pi} k_n^4 p_n^2 (2\pi) \int_{-1}^{+1} \left(1-\mu^2\right) d\mu \\ &= \frac{4}{3} c k_n^4 p_n^2 \end{aligned}$$

For the quadrupole,

$$\frac{dP}{d\Omega} \propto \left| \hat{\mathbf{r}} \times \vec{Q}_n (\hat{\mathbf{r}}) \right|^2$$

which can be quite complicated.

Suppose a charge q oscillates along the $z-$axis from $z=0$ to $z=a$,with period T. The dipole moment is:

$$p_z = zq = q\frac{a}{2}(1+\cos\omega t) = q\frac{a}{4}\left(2+e^{i\omega t}+e^{-i\omega t}\right)$$

and the quadrupole moments are

$$\begin{aligned} Q_{11} &= Q_{22} = -z^2 q = -q\frac{a^2}{4}(1+\cos\omega t)^2 \\ &= -q\frac{a^2}{4}\left(1+2\cos\omega t+\cos^2\omega t\right) \\ &= -q\frac{a^2}{4}\left(1+2\cos\omega t+\frac{\cos 2\omega t+1}{2}\right) \\ &= -q\frac{a^2}{4}\left(\frac{3}{2}+e^{i\omega t}+e^{-i\omega t}+\frac{e^{i2\omega t}+e^{-i2\omega t}}{4}\right) \end{aligned}$$

$$Q_{33} = 2z^2 q = q\frac{a^2}{2}\left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4}\right)$$

The dipole power radiated is

$$\frac{dP_n}{d\Omega} = \frac{c}{2\pi} k_n^4 p_n^2$$

and is all at the fundamental $\omega = \frac{2\pi}{T}$. We have $p_1 = qa/4$,so

$$\frac{dP_1}{d\Omega} = \frac{c}{2\pi}\left(\frac{\omega}{c}\right)^4 \left(\frac{qa}{4}\right)^2 \sin^2\theta = \frac{q^2 a^2 \omega^4}{32\pi c^3}\sin^2\theta$$

For the quadrupole, we first evaluate the vector $\vec{Q}(\hat{\mathbf{r}})$

$$Q_i = \sum_j Q_{ij} r_j$$

Thus

$$Q_1 = Q_{11}\sin\theta\cos\phi = -q\frac{a^2}{4}\left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4}\right)\sin\theta\cos\phi$$

$$Q_2 = Q_{22}\sin\theta\sin\phi = -q\frac{a^2}{4}\left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4}\right)\sin\theta\sin\phi$$

and

$$Q_3 = Q_{33}\cos\theta = q\frac{a^2}{2}\left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4}\right)\cos\theta$$

Then

$$\begin{aligned}
\hat{\mathbf{r}} \times \vec{Q} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \times (\sin\theta\cos\phi, \sin\theta\sin\phi, -2\cos\theta)\, Q_{11} \\
&= (3\sin\theta\cos\theta\sin\phi, -3\cos\theta\sin\theta\cos\phi, 0)\, q\frac{a^2}{4}\left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4}\right) \\
&= \left(\frac{3}{2}\sin 2\theta\sin\phi, -\frac{3}{2}\sin 2\theta\cos\phi, 0\right) q\frac{a^2}{4}\left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4}\right)
\end{aligned}$$

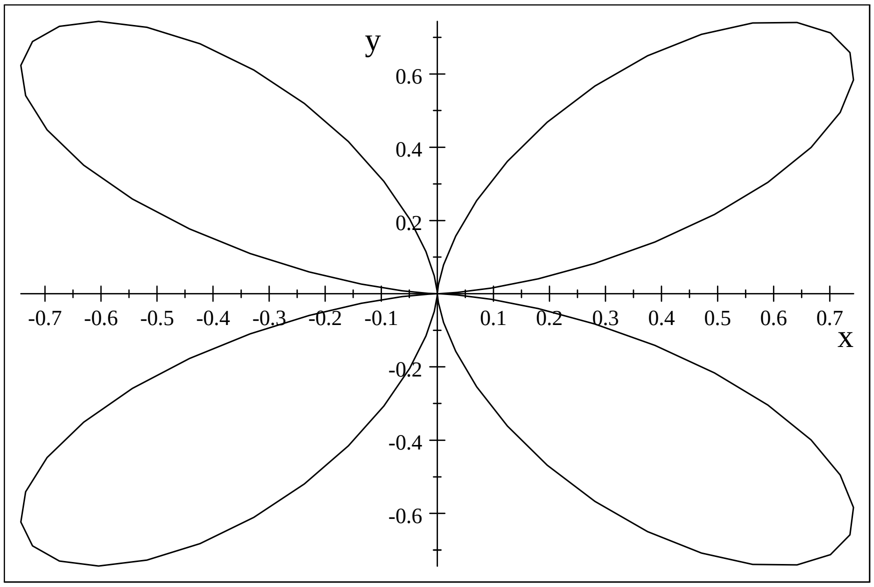
This vector has components at both ω and 2ω.

$$\begin{aligned}
\frac{dP_1}{d\Omega} &= \frac{c}{72\pi} k^6 \left|\hat{\mathbf{r}} \times \vec{Q}_1(\hat{\mathbf{r}})\right|^2 \\
&= \frac{c}{72\pi} k^6 \left(q\frac{a^2}{4}\right)^2 \left(\frac{9}{4}\sin^2 2\theta\right) \\
&= \frac{\omega^6 q^2 a^4}{512 c^5 \pi}\sin^2 2\theta
\end{aligned}$$

while at the second harmonic:

$$\begin{aligned}
\frac{dP_2}{d\Omega} &= \frac{c}{72\pi} k^6 \left(q\frac{a^2}{16}\right)^2 \left(\frac{9}{4}\sin^2 2\theta\right) \\
&= \frac{c}{72\pi}\left(\frac{2\omega}{c}\right)^6 \left(q\frac{a^2}{16}\right)^2 \left(\frac{9}{4}\sin^2 2\theta\right) \\
&= \frac{\omega^6 q^2 a^4}{128 c^5 \pi}\sin^2 2\theta \qquad (9)
\end{aligned}$$

The distribution given in equation (9) is shown below:



The total power radiated at ω is the sum of the dipole and quadrupole terms:

$$\begin{aligned}\frac{dP_1}{d\Omega} &= \frac{q^2a^2\omega^4}{32\pi c^3}\sin^2\theta + \frac{\omega^6 q^2 a^4}{2048\pi c^5}\sin^2 2\theta \\ &= \frac{q^2a^2\omega^4}{2048c^3\pi}\left(64\sin^2\theta + \frac{\omega^2a^2}{c^2}\sin^2 2\theta\right)\end{aligned}$$

The total power radiated in the quadrupole is:

$$P_1 = \int \frac{dP}{d\Omega}d\Omega = \frac{c}{72\pi}k^6 \int \left|\hat{\mathbf{r}} \times \vec{Q}_n\left(\hat{\mathbf{r}}\right)\right|^2 d\Omega$$

where

$$\left(\hat{\mathbf{r}} \times \vec{Q}_n\left(\hat{\mathbf{r}}\right)\right)_i = \varepsilon_{ijk} r_j Q_{kl} r_l$$

and thus

$$\begin{aligned}\left|\hat{\mathbf{r}} \times \vec{Q}_n\left(\hat{\mathbf{r}}\right)\right|^2 &= \varepsilon_{ijk} r_j Q_{kl} r_l \varepsilon_{imn} r_m Q^*_{np} r_p \\ &= \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) r_j Q_{kl} r_l r_m Q^*_{np} r_p \\ &= r_m r_m Q_{kl} Q^*_{kp} r_l r_p - r_j r_p Q_{ml} Q^*_{jp} r_l r_m \\ &= r_l r_p \left(Q_{kl}Q^*_{kp} - r_j r_m Q_{ml} Q^*_{jp}\right) \\ &= \vec{Q}\cdot\vec{Q}^* - \left|\hat{\mathbf{r}}\cdot\vec{Q}\right|^2\end{aligned}$$

The angle integrals give:

$$\int r_l r_p \delta\Omega = \begin{cases} 0 & \text{if} \quad l \text{ or } p = 1,2 \text{ and } l \neq p \\ \frac{4}{3}\pi & \text{if} \quad l \text{ or } p = 1,2 \text{ and } l = p \\ 2\pi\frac{2}{3} & \text{if} \quad l = p = 3 \end{cases} = \frac{4\pi}{3}\delta_{lp}$$

while for

$$\int r_l r_p r_j r_m d\Omega$$

we note that if any of the indices $l, p, j, m = 1$ or 2, then one more of them must also equal that same index to survive the integration over ϕ. We need the results

$$\begin{aligned}
\int_0^{2\pi} \cos\phi d\phi &= \int_0^{2\pi} \sin\phi d\phi = 0 \\
\int_0^{2\pi} \cos^2\phi d\phi &= \int_0^{2\pi} \sin^2\phi d\phi = \pi \\
\int_0^{2\pi} \cos^3\phi d\phi &= \int_0^{2\pi} \sin^3\phi d\phi = 0 \\
\int_0^{2\pi} \cos^4\phi d\phi &= \int_0^{2\pi} \sin^4\phi d\phi = \frac{3}{4}\pi \\
\int_0^{2\pi} \sin^2\phi \cos^2\phi d\phi &= \frac{\pi}{4}
\end{aligned}$$

Each of the components of $\hat{\mathbf{r}}$ also contains either $\sin\theta$ or cosθ, and since we need an even number of sinϕ and $\cos\phi$ terms, we never have an odd power of sinθ,so we need:

$$\begin{aligned}
\int_{-1}^{+1} \mu^4 d\mu &= \frac{2}{5} \\
\int_{-1}^{+1} \mu^2 \left(1-\mu^2\right) d\mu &= \frac{\mu^3}{3} - \frac{\mu^5}{5}\bigg|_{-1}^{+1} = \frac{4}{15} \\
\int_{-1}^{+1} \left(1-\mu^2\right)^2 d\mu &= \frac{16}{15}
\end{aligned}$$

These couple as follows:

$$\int r_l r_p r_j r_m d\Omega = \begin{cases} \pi\frac{4}{15} & \text{if} \quad \text{2 of indices } = 1,2 \text{ and 2 indices} = 3 \\ 2\pi\frac{2}{5} & \text{if} \quad \text{all indices =3} \\ \frac{\pi}{4}\frac{16}{15} & \text{if} \quad \text{2 indices } = 1 \text{ and 2 indices } = 2 \\ \frac{3\pi}{4}\frac{16}{15} & \text{if} \quad \text{all indices } = 1 \text{ or 2.} \end{cases}$$

Thus we may write:

$$\int r_l r_p r_j r_m d\Omega = \frac{4\pi}{15}\left(\delta_{lp}\delta_{jm} + \delta_{lj}\delta_{pm} + \delta_{lm}\delta_{jp}\right)$$

So we have

$$\begin{aligned}
< \quad P_n > \quad &= \frac{c}{72\pi}k^6 \int r_l r_p \left(Q_{kl}Q_{kp}^* - r_j r_m Q_{ml} Q_{jp}^*\right) d\Omega \\
&= \frac{c}{72\pi}k^6 \left[Q_{kl}Q_{kp}^* \frac{4\pi}{3}\delta_{lp} - Q_{ml}Q_{jp}^* \frac{4\pi}{15}\left(\delta_{lp}\delta_{jm} + \delta_{lj}\delta_{pm} + \delta_{lm}\delta_{jp}\right)\right] \\
&= \frac{c}{72\pi}k^6 \frac{4\pi}{3}\left[Q_{kp}Q_{kp}^* - \frac{Q_{mp}Q_{mp}^*}{5} - \frac{Q_{mj}Q_{jm}^*}{5} - \frac{Q_{mm}Q_{pp}^*}{5}\right]
\end{aligned}$$

But Q_{ij} is traceless, so the last term is zero, so

$$\begin{aligned} < \quad P_n > \ &= \frac{\omega_n^6}{54c^5}\frac{3}{5}Q_{kp}Q^*_{kp} \\ = \quad & \frac{\omega_n^6}{90c^5}\sum_{k,p}|Q_{kp}|^2 \end{aligned} \tag{10}$$

Let's recalculate the power radiated by our oscillating charge. The quadrupole term at 2ω gives

$$\begin{aligned} < \quad P_2 > \ &= \frac{(2\omega)^6}{90c^5}\left(q\frac{a^2}{16}\right)^2[1+1+4] \\ = \quad & \frac{1}{60}\frac{\omega^6}{c^5}q^2a^4 \end{aligned}$$

Now compare with the integral of our previous term (9)

$$\begin{aligned} < \quad P_2 > \ &= \int \frac{\omega^6q^2a^4}{128c^5\pi}\sin^2 2\theta d\Omega \\ = \quad & \frac{\omega^6q^2a^4}{128c^5\pi}(2\pi)\int_{-1}^{+1}4\left(1-\mu^2\right)\mu^2d\mu \\ = \quad & \frac{\omega^6q^2a^4}{128c^5\pi}(2\pi)(4)\frac{4}{15} \\ = \quad & \frac{1}{60}\frac{\omega^6}{c^5}q^2a^4 \end{aligned}$$

while at the fundamental

$$< P_1 > \ = \int \frac{\omega^6q^2a^4}{512c^5\pi}\sin^2 2\theta d\Omega = \frac{\omega^6q^2a^4}{512c^5\pi}\frac{32\pi}{15} = \frac{1}{240}\frac{\omega^6}{c^5}q^2a^4$$

and the total power formula gives:

$$< P_1 > \ = \frac{\omega^6}{90c^5}\left(q\frac{a^2}{4}\right)^2(1+1+4) = \frac{1}{240}\frac{\omega^6}{c^5}q^2a^4$$

These expressions agree.