Multipole Fields II

1 Solution of Helmholtz equation in spherical coordinates

We want to find exact solutions of the wave equation

$$\nabla^2 \psi(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

where ψ may be the electric potential Φ,or a component of $\vec{A}$, $\vec{E}$, or $\vec{B}$ outside the source region. Taking the Fourier time transform, we obtain the Helmholtz equation:

$$\left(\nabla^2 + k^2\right)\psi(\vec{x},\omega) = 0$$

In spherical coords, we have:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \nabla^2_{\text{ang}}\psi + k^2\psi = 0$$

This equation is very similar to Laplace's equation, differing only by the term in k^2. We solve it similarly, by separating variables. The solutions are of the form:

$$\psi = \sum_{l,m} f_{lm}(r)\, Y_{lm}(\theta,\phi)$$

where the functions f_{lm} satisfy the equation:

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} + k^2 f - \frac{l(l+1)}{r^2} f = 0 \tag{1}$$

where

$$\nabla^2_{\text{ang}} f_{lm} Y_{lm} = -\frac{l(l+1)}{r^2} f_{lm} Y_{lm}$$

Equation (1) shows that f_{lm} will depend on l but not on m, so henceforth we shall call it f_l. Now let

$$f_l(r) = \frac{u_l(r)}{\sqrt{r}}$$

Then

$$\frac{df}{dr} = -\frac{u_l}{2r^{3/2}} + \frac{1}{r^{1/2}}\frac{du_l}{dr}$$

and

$$\frac{d^2 f}{dr^2} = \frac{3u_l}{4r^{5/2}} - \frac{1}{r^{3/2}}\frac{du_l}{dr} + \frac{1}{r^{1/2}}\frac{d^2 u_l}{dr^2}$$

Thus equation (1) becomes:

$$\begin{aligned}
\frac{3u_l}{4r^{5/2}} - \frac{1}{r^{3/2}}\frac{du_l}{dr} + \frac{1}{r^{1/2}}\frac{d^2 u_l}{dr^2} + \frac{2}{r}\left(-\frac{u_l}{2r^{3/2}} + \frac{1}{r^{1/2}}\frac{du_l}{dr}\right) + \left[k^2 - \frac{l(l+1)}{r^2}\right]\frac{u_l(r)}{\sqrt{r}} &= 0 \\
\frac{1}{r^{1/2}}\frac{d^2 u_l}{dr^2} + \frac{1}{r^{3/2}}\frac{du_l}{dr} + \left[k^2 - \frac{l(l+1)+1/4}{r^2}\right]\frac{u_l(r)}{\sqrt{r}} &= 0
\end{aligned}$$

Now we multiply by $\sqrt{r}$ to get:

$$\frac{d^2u_l}{dr^2}+\frac{1}{r}\frac{du_l}{dr}+\left[k^2-\frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right]u_l\left(r\right)=0$$

which is Bessel's equation with $\nu=l+1/2$. Thus the solutions for f_l are of the form

$$\frac{J_{l+1/2}}{\sqrt{r}}$$

These are the *spherical Bessel functions:*

$$j_l\left(x\right)=\sqrt{\frac{\pi}{2}}\frac{J_{l+1/2}\left(x\right)}{\sqrt{x}}$$

The other Bessel functions are defined similarly. Thus:

$$n_l\left(x\right)=\sqrt{\frac{\pi}{2}}\frac{N_{l+1/2}\left(x\right)}{\sqrt{x}}$$

and

$$\begin{aligned}h_l^{(1,2)}\left(x\right) &= \sqrt{\frac{\pi}{2x}}\left(J_{l+1/2}\left(x\right)\pm iN_{l+1/2}\left(x\right)\right)\\ &= j_l\left(x\right)\pm in_l\left(x\right)\end{aligned}$$

The hs are particularly useful since they have wave-like behavior at large argument:

$$h_l^{(1)}\left(x\right)\simeq\left(-i\right)^{l+1}\frac{e^{ix}}{x}\text{ for }x\gg l$$

The j_l are well behaved at both small and large arguments:

$$j_l\left(x\right)\simeq\frac{x^l}{\left(2l+1\right)!}\text{ for }x\ll 1,l$$

and

$$j_l\left(x\right)\simeq\frac{\sin\left(x-l\frac{\pi}{2}\right)}{x}\text{ for }x\gg l$$

2 The Green's function

Next we expand the Green's function for the Helmholtz equation in these eigenfunctions. The appropriate equation is

$$\left(\nabla^2+k^2\right)G\left(\vec{x},\vec{x}',\omega\right)=-\delta\left(\vec{x}-\vec{x}'\right)$$

Expanding G in spherical harmonics, we have:

$$G\left(\vec{x},\vec{x}',\omega\right)=\sum_{l,m}g_l\left(r,r'\right)Y_{lm}\left(\theta,\phi\right)Y_{lm}^*\left(\theta',\phi'\right)$$

Stuffing in:

$$\sum_{l,m} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_l}{dr}\right)+\left[k^2-\frac{l\left(l+1\right)}{r^2}\right]g_l Y_{lm}\left(\theta,\phi\right)Y^*_{lm}\left(\theta',\phi'\right)=-\delta\left(\vec{x}-\vec{x}'\right)=-\frac{\delta\left(r-r'\right)}{r^2}\delta\left(\mu-\mu'\right)\delta\left(\phi-\phi'\right)$$

Now multiply both sides by $Y^*_{lm}\left(\theta,\phi\right)$ and integrate over the angles. We get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_l}{dr}\right)+\left[k^2-\frac{l\left(l+1\right)}{r^2}\right]g_l Y^*_{lm}\left(\theta',\phi'\right)=-\frac{\delta\left(r-r'\right)}{r^2}Y^*_{lm}\left(\theta',\phi'\right)$$

so the equation for g_l is:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_l}{dr}\right)+\left[k^2-\frac{l\left(l+1\right)}{r^2}\right]g_l=-\frac{\delta\left(r-r'\right)}{r^2}$$

For $r<r'$ we need a solution finite at $r=0$: the j_l. We want an outgoing wave at infinity, so we choose $h_l^{(1)}$in the region $r>r'$. Then by the symmetry of the Green's function, we get:

$$g_l\left(r,r'\right)=A_l j_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right)$$

Next we multiply by r^2 and integrate across the boundary at $r=r'$:

$$r^2\frac{dg_l}{dr}\bigg|_{r'-\varepsilon}^{r'+\varepsilon}=-1$$

$$kA_l\left(h_l^{(1)\prime}\left(kr'\right)j_l\left(kr'\right)-j_l'\left(kr'\right)h_l^{(1)}\left(kr'\right)\right)=-\frac{1}{\left(r'\right)^2}$$

The term in parentheses on the left hand side is the Wronskian of the two solutions j_l and $h_l^{(1)}$. Using the large argument form of these functions, we get:

$$\begin{aligned}
W &= \left(-i\right)^{l+1}\frac{e^{ix}}{x}\left(i-\frac{1}{x}\right)\frac{\sin\left(x-l\frac{\pi}{2}\right)}{x}-\left(-i\right)^{l+1}\frac{e^{ix}}{x}\left[\frac{\cos\left(x-l\frac{\pi}{2}\right)}{x}-\frac{\sin\left(x-l\frac{\pi}{2}\right)}{x^2}\right]\\
&= \left(-i\right)^{l+1}\frac{e^{ix}}{x^2}\left(\left[i\sin\left(x-l\frac{\pi}{2}\right)-\frac{\sin\left(x-l\frac{\pi}{2}\right)}{x}-\cos\left(x-l\frac{\pi}{2}\right)+\frac{\sin\left(x-l\frac{\pi}{2}\right)}{x}\right]\right)\\
&= \left(-i\right)^{l+1}\frac{e^{ix}}{x^2}\left(-e^{-i\left(x-l\frac{\pi}{2}\right)}\right)=\frac{\left(-i\right)^{l+1}\left(-i^l\right)}{x^2}=\frac{\left(-1\right)^{l+2}i^{2l+1}}{x^2}=\frac{\left(-1\right)^l\left(-1\right)^l i}{x^2}\\
&= \frac{i}{x^2}
\end{aligned}$$

Thus

$$kA_l\frac{i}{\left(kr'\right)^2}=-\frac{1}{\left(r'\right)^2}$$

and so

$$A_l=-\frac{k}{i}=ik$$

and thus

$$G\left(\vec{x},\vec{x}'\right)=\frac{e^{ikR}}{4\pi R}=ik\sum_{l,m}j_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right)Y_{lm}\left(\theta,\phi\right)Y^*_{lm}\left(\theta',\phi'\right)\qquad(2)$$

3 Solution to the vector Helmholtz equation

The equation satisfied by the electric and magnetic fields is:

$$\left(\nabla^2 + k^2\right)\vec{E}=0$$

In any coordinate system other than Cartesian, we cannot separate this equation into components because the unit vectors are not constants. Thus we need to find a scalar quantity that satisfies the Helmholtz equation. Let's investigate

$$\begin{aligned}
\nabla^2\left(\vec{r}\cdot\vec{E}\right) &= \partial_i\partial^i r_j E^j = \partial_i\left(\left(\partial^i r_j\right)E^j + r_j\partial^i E^j\right)\\
&= \left(\partial_i\partial^i r_j\right)E^j + \left(\partial^i r_j\right)\partial_i E^j + \left(\partial_i r_j\right)\left(\partial^i E^j\right) + r_j\partial_i\partial^i E^j\\
&= 0 + \delta^i_j\partial_i E^j + \delta_{ij}\partial^i E^j + r_j\nabla^2 E^j\\
&= 2\left(\vec{\nabla}\cdot\vec{E}\right) + \vec{r}\cdot\nabla^2\vec{E}
\end{aligned}$$

But outside the source, $\vec{\nabla}\cdot\vec{E}=0$, and so

$$\nabla^2\left(\vec{r}\cdot\vec{E}\right) = \vec{r}\cdot\nabla^2\vec{E}$$

Thus we have

$$\left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{E}\right) = \vec{r}\cdot\left(\nabla^2 + k^2\right)\vec{E}=0$$

and thus $\vec{r}\cdot\vec{E}$ is the scalar quantity we need.

Notice we can do the same thing with $\vec{B}$, since $\vec{\nabla}\cdot\vec{B} = 0$ always. The solutions for $\vec{r}\cdot\vec{E}$ and $\vec{r}\cdot\vec{B}$ will thus be of the form:

$$\vec{r}\cdot\vec{E}= -\frac{l\left(l+1\right)}{k}g_l\left(kr\right)Y_{lm}\left(\theta,\phi\right)$$

where the constant $-\frac{l(l+1)}{k}$ has been inserted for future convenience, and the function g_l must be a combination of the spherical Bessel functions, for example

$$g_l = A_l^{(1)}h_l^{(1)}\left(kr\right) + A_l^{(2)}h_l^{(2)}\left(kr\right)$$

Then the corresponding $\vec{B}$ will be given by:

$$\vec{B} = -\frac{i}{k}\vec{\nabla}\times\vec{E} = \frac{1}{ik}\vec{\nabla}\times\vec{E}$$

We can borrow some ideas from quantum mechanics. The angular momentum operator is:

$$\vec{L} = \frac{1}{i}\vec{r}\times\vec{\nabla}$$

and its eigenfunctions are the Y_{lm}. Notice that

$$\vec{r}\cdot\vec{L}=0$$

Then

$$k\vec{r}\cdot\vec{B} = \frac{1}{i}\vec{r}\cdot\left(\vec{\nabla}\times\vec{E}\right) = \frac{1}{i}\left(\vec{r}\times\vec{\nabla}\right)\cdot\vec{E} = \vec{L}\cdot\vec{E} \qquad (3)$$

Thus we may define the following eigenfunctions for the problem:

The *electric* or *transverse magnetic* mode has $\vec{B}$ purely tangential:

$$\begin{aligned}
\vec{r}\cdot\vec{E}_{lm}^{(E)} &= -\frac{l\,(l+1)}{k}g_l\,(kr)\,Y_{lm}\,(\theta,\phi)\\
\vec{r}\cdot\vec{B}_{lm}^{(E)} &= 0
\end{aligned}$$

The *magnetic* or *transverse electric* mode has $\vec{E}$ purely tangential:

$$\begin{aligned}
\vec{r}\cdot\vec{B}_{lm}^{(M)} &= \frac{l\,(l+1)}{k}g_l\,(kr)\,Y_{lm}\,(\theta,\phi)\\
\vec{r}\cdot\vec{E}_{lm}^{(M)} &= 0
\end{aligned}$$

Thus from equation (3),

$$\vec{L}\cdot\vec{E}_{lm}^{(M)} = l\,(l+1)\,g_l\,(kr)\,Y_{lm}\,(\theta,\phi) \tag{4}$$

4 Properties of $\tilde{\mathbf{L}}$

$$\begin{aligned}
\vec{L} &= \frac{1}{i}\vec{r}\times\vec{\nabla}\\
&= \frac{1}{i}\left\{\hat{\mathbf{x}}\,(y\partial_z - z\partial_y) + \hat{\mathbf{y}}\,(z\partial_x - x\partial_z) + \hat{\mathbf{z}}\,(x\partial_y - y\partial_x)\right\}
\end{aligned}$$

Now we want to write everything in spherical coordinates while keeping the Cartesian components:

$$\begin{aligned}
\vec{\nabla}_{\text{cartesian}} &= \frac{\partial x^i_{sph}}{\partial x^j_{cart}}\partial_{j\text{spherical}}\\
&= \begin{pmatrix} \sin\theta\cos\phi & \frac{\cos\theta\cos\phi}{r} & -\frac{\sin\phi}{r\sin\theta}\\ \sin\theta\sin\phi & \frac{\cos\theta\sin\phi}{r} & \frac{\cos\phi}{r\sin\theta}\\ \cos\theta & -\frac{\sin\theta}{r} & 0 \end{pmatrix}\vec{\nabla}_{\text{spherical}}
\end{aligned}$$

Thus

$$\begin{aligned}
iL_x &= r\sin\theta\sin\phi\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right) - r\cos\theta\left(\sin\theta\sin\phi\partial_r + \frac{\cos\theta\sin\phi}{r}\partial_\theta + \frac{\cos\phi}{r\sin\theta}\partial_\phi\right)\\
&= -\sin\phi\partial_\theta - \cos\phi\cot\theta\partial_\phi
\end{aligned}$$

$$\begin{aligned}
iL_y &= r\cos\theta\left(\sin\theta\cos\phi\partial_r - \frac{\cos\theta\cos\phi}{r}\partial_\theta - \frac{\sin\phi}{r\sin\theta}\partial_\phi\right) - r\sin\theta\cos\phi\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\\
&= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi
\end{aligned}$$

$$\begin{aligned}
iL_z = {} & r\sin\theta\cos\phi\left(\sin\theta\sin\phi\partial_r + \frac{\cos\theta\sin\phi}{r}\partial_\theta + \frac{\cos\phi}{r\sin\theta}\partial_\phi\right)\\
& -r\sin\theta\sin\phi\left(\sin\theta\cos\phi\partial_r + \frac{\cos\theta\cos\phi}{r}\partial_\theta - \frac{\sin\phi}{r\sin\theta}\partial_\phi\right) = \partial_\phi
\end{aligned}$$

L_z is very nice. We can form linear combinations of L_x and L_y that look almost as nice:

$$\begin{aligned}
L_{\pm} &= L_x \pm i L_y \\
&= \frac{1}{i}\left(-\sin\phi\partial_\theta - \cos\phi\cot\theta\partial_\phi \pm i\left(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\right)\right) \\
&= \left(\pm\cos\phi\partial_\theta + i\sin\phi\partial_\theta + i\cos\phi\cot\theta\partial_\phi \mp \cot\theta\sin\phi\partial_\phi\right) \\
&= e^{\pm i\phi}\left(\pm\partial_\theta + i\cot\theta\partial_\phi\right)
\end{aligned}$$

Then

$$\begin{aligned}
\vec{L}\cdot\vec{L} &= L_x^2 + L_y^2 + L_z^2 \\
&= -\left(\sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi\right)^2 - \left(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\right)^2 - \partial_\phi^2 \\
&= -\partial_\theta^2 - \sin\phi\cos\phi\partial_\theta\cot\theta\partial_\phi - \cos^2\phi\cot\theta\partial_\theta - \cos^2\phi\cot^2\theta\partial_\phi^2 \\
&\quad + \cos\phi\sin\phi\partial_\theta\cot\theta\partial_\phi - \cot\theta\sin\phi^2\partial_\theta - \cot^2\theta\sin^2\phi\partial_\phi^2 - \partial_\phi^2 \\
&= -\partial_\theta^2 - \cot\theta\partial_\theta - \left(\cot^2\theta + 1\right)\partial_\phi^2 \\
&= -\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) - \frac{1}{\sin^2\theta}\partial_\phi^2
\end{aligned} \tag{5}$$

which is the angular part of ∇^2. Thus:

$$L^2 Y_{lm} = l\left(l+1\right)Y_{lm}$$

We also have

$$L_z Y_{lm} = m Y_{lm}$$

5 Complete solution for $\tilde{\text{E}}$.

Suppose

$$\vec{E} = \vec{L} g_l\left(kr\right) Y_{lm}\left(\theta,\phi\right)$$

Then

$$\vec{L}\cdot\vec{E} = g_l L^2 Y_{lm} = l\left(l+1\right) g_l Y_{lm}$$

which is exactly what we need (cf equation 4). Thus the solutions are:

Magnetic or transverse electric modes:

$$\begin{aligned}
\vec{E}_{lm}^{(M)} &= g_l\left(kr\right)\vec{L} Y_{lm}\left(\theta,\phi\right) \\
\vec{B}_{lm}^{(M)} &= -\frac{i}{k}\vec{\nabla}\times\vec{E}_{lm}^{(M)}
\end{aligned} \tag{6}$$

and

Electric or transverse magnetic modes:

$$\begin{aligned}
\vec{B}_{lm}^{(E)} &= g_l\left(kr\right)\vec{L} Y_{lm}\left(\theta,\phi\right) \\
\vec{E}_{lm}^{(E)} &= \frac{i}{k}\vec{\nabla}\times\vec{B}_{lm}^{(E)}
\end{aligned} \tag{7}$$

Now we define the *vector spherical harmonics:*

$$\vec{X}_{lm} = \frac{1}{\sqrt{l\,(l+1)}}\vec{L}Y_{lm}$$

so that

$$\vec{E}_{lm}^{(M)} = g_l\,(kr)\,\vec{X}_{lm}$$

which are the appropriate eigenfunctions for the problem.

Combining the modes we get the general solution:

$$\begin{aligned} \vec{E} &= \sum_{l,m}\left[a_M\,(l,m)\,g_l\,(kr)\,\vec{X}_{lm} + \frac{i}{k}a_E\,(l.m)\,\vec{\nabla}\times f_l\,(kr)\,\vec{X}_{lm}\right] \\ \vec{B} &= \sum_{l,m}\left[a_E\,(l,m)\,f_l\,(kr)\,\vec{X}_{lm} - \frac{i}{k}a_M\,(l.m)\,\vec{\nabla}\times g_l\,(kr)\,\vec{X}_{lm}\right] \end{aligned} \tag{8}$$

6 Finding the coefficients

Note that, since $\vec{r}\cdot\vec{X}_{lm} = 0$,then

$$\begin{aligned} \vec{r}\cdot\vec{B} &= \sum_{l,m} -\frac{i}{k}a_M\,(l.m)\,\vec{r}\cdot\vec{\nabla}\times\left(g_l\,(kr)\,\vec{X}_{lm}\right) \\ &= \sum_{l,m} -\frac{1}{k\sqrt{l\,(l+1)}}a_M\,(l.m)\,\vec{r}\cdot\left(\vec{\nabla}g_l\times\left(\vec{r}\times\vec{\nabla}\right)Y_{lm} + g_l\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right)Y_{lm}\right) \end{aligned}$$

But $\vec{\nabla}g_l$ is in the $\hat{\mathbf{r}}$ direction, and so $\vec{\nabla}g_l\times\left(\vec{r}\times\vec{\nabla}\right)Y_{lm}$ is perpendicular to $\hat{\mathbf{r}}$, and so its dot product with $\vec{r}$ is zero. Also:

$$\begin{aligned} i\vec{\nabla}\times\vec{L} &= \vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right) = \partial_j\left(r^i\partial^j\right) - \partial^j\left(r_j\partial^i\right) \\ &= \delta^i_j\partial^j + r^i\nabla^2 - 3\partial^i - r_j\partial^i\partial^j = -r^i\nabla^2 - 2\partial^i - r_j\partial^i\partial^j \end{aligned}$$

and

$$\begin{aligned} \vec{\nabla}\left(1+\vec{r}\cdot\vec{\nabla}\right) &= \partial^i\left(1+r_j\partial^j\right) = \partial^i + \delta^i_j\partial^j + r_j\partial^i\partial^j \\ &= 2\partial^i + r_j\partial^i\partial^j \end{aligned}$$

Thus: Thus:

$$i\vec{\nabla}\times\vec{L}{=}\vec{r}\nabla^2 - \vec{\nabla}\left(1+\vec{r}\cdot\vec{\nabla}\right) \tag{9}$$

$$
\begin{aligned}
\vec{r}\cdot\vec{B} &= \sum_{l,m} -\frac{1}{k\sqrt{l(l+1)}} a_M(l.m)\, g_l(kr)\, \vec{r}\cdot\left(\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right)Y_{lm}\right) \\
&= \sum_{l,m} -\frac{1}{k\sqrt{l(l+1)}} a_M(l.m)\, g_l(kr)\, \vec{r}\cdot\left(\vec{r}\nabla^2-\vec{\nabla}\left(1+\vec{r}\cdot\vec{\nabla}\right)Y_{lm}\right) \\
&= \sum_{l,m} -\frac{1}{k\sqrt{l(l+1)}} a_M(l.m)\, g_l(kr)\left(r^2\nabla^2-\vec{r}\cdot\vec{\nabla}\left(1+\vec{r}\cdot\vec{\nabla}\right)Y_{lm}\right)
\end{aligned}
$$

But Y_{lm} is independent of r, so $\vec{r}\cdot\vec{\nabla}Y_{lm}=0$, and we already know $\nabla^2 Y_{lm}$:

$$
\begin{aligned}
\vec{r}\cdot\vec{B} &= \sum_{l,m} -\frac{1}{k\sqrt{l(l+1)}} a_M(l.m)\, g_l(kr)\, r^2\frac{-l(l+1)}{r^2}Y_{lm} \\
&= \sum_{l,m} \frac{\sqrt{l(l+1)}}{k} a_M(l.m)\, g_l(kr)\, Y_{lm} \qquad (10)
\end{aligned}
$$

Then using the orthogonality of the Y_{lm},we have:

$$
a_M(l.m)\, g_l(kr) = \frac{k}{\sqrt{l(l+1)}}\int \tilde{\mathbf{r}}\cdot\tilde{\mathbf{B}}Y_{lm}^{*}d \qquad (11)
$$

Similarly we find:

$$
a_E(l.m)\, f_l(kr) = \frac{-k}{\sqrt{l(l+1)}}\int \vec{r}\cdot\vec{E}Y_{lm}^{*}d
$$

7 Properties of the fields

7.1 The near zone

For $kr \ll 1$, we may use the small argument expansion of the Bessel functions:

$$
j_l \sim (kr)^l
$$

while

$$
n_l \sim -\frac{(2l-1)!!}{x^{l+1}}
$$

Thus both hs behave like ns for $kr \ll 1$. Then for the electric mode we have:

$$
\vec{B}_{lm}^{(E)} \sim \vec{L}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}
$$

and

$$
\vec{E}_{lm}^{(E)} = -\frac{i}{k}\vec{\nabla}\times\vec{B}_{lm}^{(E)} = -\frac{i}{k}\vec{\nabla}\times\vec{L}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}
$$

and using equation (9) we may write the electric field as:

$$
\vec{E}_{lm}^{(E)} = -\frac{1}{k}\left[\vec{r}\nabla^2-\vec{\nabla}\left(1+\vec{r}\cdot\vec{\nabla}\right)\right]\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}
$$

The first term is zero because $\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ is a solution of Laplace's equation. Thus:

$$\begin{aligned}\vec{E}^{(E)}_{lm} &= \frac{1}{k}\vec{\nabla}\left(1+\vec{r}\cdot\vec{\nabla}\right)\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\\ &= \frac{1}{k}\vec{\nabla}\left(1-(l+1)\right)\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\\ &= -\frac{l}{k}\vec{\nabla}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\end{aligned}$$

Now $\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ is a solution of Laplace's equation and thus represents an electrostatic potential: it is a static multipole potential, and its gradient therefore represents a static multipole field. Thus in the near zone the electric mode is a static electric multipole field that oscillates in time (because of the $e^{-i\omega t}$ factor that multiplies $\vec{E}(k)$). This is the origin of the name "electric" for this mode. Similarly, the magnetic field in the magnetic mode has a static multipole character in the near zone.

7.2 The far zone

When $kr \gg 1$ we may expand the Bessel functions using the large argument expansion.

$$h^{(1)}_l(kr) \sim (-i)^{l+1}\frac{e^{ikr}}{kr}$$

and so

$$\vec{B}^{(E)}_{lm} \sim (-i)^{l+1}\frac{e^{ikr}}{kr}\vec{L}Y_{lm}$$

and

$$\begin{aligned}\tilde{\mathbf{E}}^{(E)}_{lm} &\sim i\frac{(-i)^{l+1}}{k}\tilde{\boldsymbol{\nabla}}\times\frac{e^{ikr}}{kr}\tilde{\mathbf{L}}Y_{lm}\\ &= -\frac{(-i)^{l+2}}{k^2}\left(\tilde{\boldsymbol{\nabla}}\frac{e^{ikr}}{r}\times\tilde{\mathbf{L}}Y_{lm}+\frac{e^{ikr}}{r}\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{L}}Y_{lm}\right)\\ &= \frac{(-i)^{l}}{k^2}\frac{e^{ikr}}{r}\left(\left[i\tilde{\mathbf{k}}-\frac{\hat{\mathbf{r}}}{r}\right]\times\tilde{\mathbf{L}}Y_{lm}+\frac{1}{i}\left[\tilde{\mathbf{r}}\nabla^2-\tilde{\boldsymbol{\nabla}}\left(1+\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}\right)\right]Y_{lm}\right)\end{aligned}$$

where we used equation (9). Again $\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}Y_{lm}\equiv 0.$ and so dropping terms of order $1/kr$, we have:

$$\tilde{\mathbf{E}}^{(E)}_{lm} = \frac{(-i)^{l}}{k}\frac{e^{ikr}}{r}\left(i\hat{\mathbf{k}}\times\tilde{\mathbf{L}}Y_{lm}+\frac{1}{ik}\left[\tilde{\mathbf{r}}\frac{-l(l+1)}{r^2}-\tilde{\boldsymbol{\nabla}}\right]Y_{lm}\right)$$

Both terms in the square brackets are of order $1/kr$ copmared with the first term, and so we drop them. Then:

$$\begin{aligned}\tilde{\mathbf{E}}^{(E)}_{lm} &= -(-i)^{l+1}\frac{e^{ikr}}{kr}\hat{\mathbf{k}}\times\tilde{\mathbf{L}}Y_{lm}\\ &= -\hat{\mathbf{k}}\times\tilde{\mathbf{B}}^{(E)}_{lm} \qquad (12)\end{aligned}$$

and the field is a radiation field.

8 Orthogonality of the $\tilde{\mathbf{X}}_{lm}$

Our next task is to demonstrate that the $\tilde{\mathbf{X}}_{lm}$ are orthogonal, that is

$$\int_{sphere} \tilde{\mathbf{X}}_{lm} \cdot \tilde{\mathbf{X}}^*_{l'm'} d \quad = \delta_{ll'}\delta_{mm'} \tag{13}$$

The first step is to evaluate $\tilde{\mathbf{X}}_{lm}$ is terms of the Y_{lm}.

$$\tilde{\mathbf{X}}_{lm} = \frac{1}{\sqrt{l\,(l+1)}}\tilde{\mathbf{L}}Y_{lm} = \frac{1}{\sqrt{l\,(l+1)}}\left(\hat{\mathbf{x}}\left[\frac{L_+ + L_-}{2}\right] + \hat{\mathbf{y}}\left[\frac{L_+ - L_-}{2i}\right] + \hat{\mathbf{z}}L_z\right)Y_{lm}$$

So we need to evaluate L_+Y_{lm} and L_-Y_{lm}.

$$\begin{aligned}
L_\pm Y_{lm} &= e^{\pm i\phi}\left(\pm\partial_\theta + i\cot\theta\partial_\phi\right)Y_{lm} \\
&= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{\pm i\phi}\left(\pm\partial_\theta + i\cot\theta\partial_\phi\right)P_l^m(\cos\theta)\,e^{im\phi} \\
&= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left(\pm\partial_\theta - m\cot\theta\right)P_l^m(\cos\theta)\,e^{i(m\pm1)\phi}
\end{aligned}$$

Now

$$\begin{aligned}
\left(\pm\partial_\theta - m\cot\theta\right)P_l^m(\cos\theta) &= \left(\mp\sin\theta\partial_\mu - m\frac{\mu}{\sin\theta}\right)P_l^m(\mu) \\
&= \frac{1}{\sin\theta}\left(\mp\left(1-\mu^2\right)P_l^{m\prime}(\mu) - m\mu P_l^m(\mu)\right)
\end{aligned}$$

But from the m-raising and $m-$lowering recursion relations for the Legendre functions (eg Problem 8.16 pg 431 in Lea[1]),

$$\begin{aligned}
\left(1-\mu^2\right)P_l^{m\prime}(\mu) &= -m\mu P_l^m - \sqrt{1-\mu^2}P_l^{m+1} \\
&= m\mu P_l^m + (l+m)(l-m+1)\sqrt{1-\mu^2}P_l^{m-1}
\end{aligned}$$

and thus:

$$\begin{aligned}
\left(+\partial_\theta - m\cot\theta\right)P_l^m(\cos\theta) &= \frac{1}{\sin\theta}\left(-\left(1-\mu^2\right)P_l^{m\prime}(\mu) - m\mu P_l^m(\mu)\right) \\
&= \frac{1}{\sin\theta}\left(\sqrt{1-\mu^2}P_l^{m+1} + m\mu P_l^m - m\mu P_l^m(\mu)\right) \\
&= \frac{1}{\sin\theta}\sqrt{1-\mu^2}P_l^{m+1} = P_l^{m+1}
\end{aligned}$$

and so

$$L_+Y_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^{m+1}e^{i(m+1)\phi}$$

$$L_+Y_{lm} = \sqrt{\frac{(l-m)!}{(l+m)!}\frac{(l+m+1)!}{(l-m-1)!}}Y_{l,m+1}$$

[1] Note: there is some disagreement in the literature as to the definition of P_l^m. My choice is the same as Jackson's. Some authors (eg Butkov) have a definition that differs by a factor $(-1)^m$.

$$L_+ Y_{lm} = \sqrt{(l-m)(l+m+1)} Y_{l,m+1}$$

Similarly

$$\begin{aligned}
(-\partial_\theta - m\cot\theta) P_l^m(\cos\theta) &= \frac{1}{\sin\theta}\left(+\left(1-\mu^2\right) P_l^{m\prime}(\mu) - m\mu P_l^m(\mu)\right) \\
&= \frac{1}{\sin\theta}\left(m\mu P_{lm} + (l+m)(l-m+1)\sqrt{1-\mu^2} P_l^{m-1} - m\mu P_l^m(\mu)\right) \\
&= (l+m)(l-m+1) P_l^{m-1}
\end{aligned}$$

and thus

$$\begin{aligned}
L_- Y_{lm} &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} (l+m)(l-m+1) P_l^{m-1} e^{i(m-1)\phi} \\
&= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m+1)!}{(l-m+1)(l+m-1)!(l+m)}} (l+m)(l-m+1) P_l^{m-1} e^{i(m-1)\phi} \\
&= \frac{(l+m)(l-m+1)}{\sqrt{(l-m+1)(l+m)}} Y_{l,m-1} = \sqrt{(l+m)(l-m+1)} Y_{l,m-1}
\end{aligned}$$

The quantity in the square root may be put in a nicer form:

$$(l+m)(l-m+1) = l^2 + l - m^2 + m = l(l+1) - m(m-1)$$

Thus

$$L_- Y_{lm} = \sqrt{l(l+1) - m(m-1)} Y_{l,m-1}$$

and similarly

$$L_+ Y_{lm} = \sqrt{l(l+1) - m(m+1)} Y_{l,m+1}$$

Then:

$$\begin{aligned}
\vec{X}_{lm} &= \frac{1}{\sqrt{l(l+1)}}\left(\hat{\mathbf{x}}\left[\frac{L_+ + L_-}{2}\right] + \hat{\mathbf{y}}\left[\frac{L_+ - L_-}{2i}\right] + \hat{\mathbf{z}} L_z\right) Y_{lm} \\
&= \frac{1}{\sqrt{l(l+1)}}\left(\begin{array}{c} \frac{1}{2}\sqrt{l(l+1) - m(m+1)} Y_{l,m+1}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \\ +\frac{1}{2}\sqrt{l(l+1) - m(m-1)} Y_{l,m-1}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + m Y_{lm}\hat{\mathbf{z}} \end{array}\right) \qquad (14)
\end{aligned}$$

Now note that:

$$\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)\cdot\left(\frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{2}\right)^* = \frac{\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} - \hat{\mathbf{y}}\cdot\hat{\mathbf{y}}}{4} = 0$$

and similarly

$$\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)\cdot\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)^* = \frac{1}{2}$$

Then the orthogonality integral becomes:

$$\begin{aligned}
&\int_{\text{sphere}} \vec{X}_{lm}\cdot\vec{X}^*_{l'm'} d \\
&= \frac{1}{2l(l+1)}\int_{\text{sph}}\left[\begin{array}{c} \sqrt{l(l+1) - m(m+1)}\sqrt{l'(l'+1) - m'(m'+1)} Y_{l,m+1} Y^*_{l',m'+1} \\ +\sqrt{l(l+1) - m(m-1)}\sqrt{l'(l'+1) - m'(m'-1)} Y_{l,m-1} Y^*_{l',m'-1} \\ +2mm' Y_{lm} Y^{*\prime}_{lm} \end{array}\right] d
\end{aligned}$$

$$\int_{\text{sphere}} \vec{X}_{lm} \cdot \vec{X}^*_{l'm'} d \quad = \quad \frac{1}{2l\,(l+1)} \left[l\,(l+1) - m\,(m+1) + l\,(l+1) - m\,(m-1) + 2m^2 \right] \delta_{ll'} \delta_{mm'}$$

$$= \quad \frac{1}{2l\,(l+1)} 2l\,(l+1)\,\delta_{ll'}\delta_{mm'} = \delta_{ll'}\delta_{mm'}$$

where we used the orthogonality of the Y_{lm}.

Also

$$\int_{\text{sphere}} \vec{X}^*_{l'm'} \cdot \left(\vec{r} \times \vec{X}_{lm} \right) d \quad = \frac{1}{l\,(l+1)} \int_{\text{sphere}} \left(\frac{1}{i} \vec{r} \times \vec{\nabla} Y^*_{l'm'} \right) \cdot \left(\vec{r} \times \left[\frac{1}{i} \vec{r} \times \vec{\nabla} Y_{lm} \right] \right) d$$

where

$$\vec{r} \times \left[\vec{r} \times \vec{\nabla} Y_{lm} \right] = \vec{r} \left(\vec{r} \cdot \vec{\nabla} Y_{lm} \right) - r^2 \vec{\nabla} Y_{lm} = -r^2 \vec{\nabla} Y_{lm}$$

Thus:

$$\int_{\text{sphere}} \vec{X}^*_{l'm'} \cdot \left(\vec{r} \times \vec{X}_{lm} \right) d \quad = \frac{-1}{l\,(l+1)} \int_{\text{sphere}} \left(\vec{r} \times \vec{\nabla} Y^*_{l'm'} \right) \cdot \left(-r^2 \vec{\nabla} Y_{lm} \right) d$$

Now

$$\vec{\nabla} \times \left(\vec{r} Y^*_{l'm'} \right) = \vec{\nabla} Y^*_{l'm'} \times \vec{r} + Y^*_{l'm'} \vec{\nabla} \times \vec{r}$$

and $\vec{\nabla} \times \vec{r} = 0$,so:

$$\int_{\text{sphere}} \vec{X}^*_{l'm'} \cdot \left(\vec{r} \times \vec{X}_{lm} \right) d \quad = \frac{r^2}{l\,(l+1)} \int_{\text{sphere}} \left(-\vec{\nabla} \times \left(\vec{r} Y^*_{l'm'} \right) \right) \cdot \vec{\nabla} Y_{lm}\, d$$

And

$$\vec{\nabla} \cdot \left(\vec{a} \times \vec{b} \right) = \vec{b} \cdot \left(\vec{\nabla} \times \vec{a} \right) - \vec{a} \cdot \left(\vec{\nabla} \times \vec{b} \right)$$

so taking $\vec{a} = \vec{\nabla} Y_{lm}$ and $\vec{b} = \vec{r} Y^*_{l'm'}$, we have $\vec{\nabla} \times \vec{a} = 0$,and then

$$\int_{\text{sphere}} \vec{X}^*_{l'm'} \cdot \left(\vec{r} \times \vec{X}_{lm} \right) d \quad = \frac{r^2}{l\,(l+1)} \int_{\text{sphere}} \vec{\nabla} \cdot \left(\vec{\nabla} Y_{lm} \times \vec{r} Y^*_{l'm'} \right) d$$

Then

$$\int_{r_1}^{r_2} \int_{\text{sphere}} \vec{X}^*_{l'm'} \cdot \left(\vec{r} \times \vec{X}_{lm} \right) d \quad dr \quad = \quad \frac{1}{l\,(l+1)} \int_{\text{spherical shell}} \vec{\nabla} \cdot \left(\vec{\nabla} Y_{lm} \times \vec{r} Y^*_{l'm'} \right) dV$$

$$= \quad \frac{1}{l\,(l+1)} \int_{\text{spherical shell}} \hat{\mathbf{r}} \cdot \left(\vec{\nabla} Y_{lm} \times \vec{r} Y^*_{l'm'} \right) dA$$

Here the integrand is zero, because $\vec{\nabla} Y_{lm}$ is tangential, and therefore so is $\vec{\nabla} Y_{lm} \times \vec{r}$, and thus its dot product with $\hat{\mathbf{r}}$ is zero. Thus the volume integral is zero for arbitrary values of r_1 and r_2. So we must conclude that

$$\int_{\text{sphere}} \vec{X}^*_{l'm'} \cdot \left(\vec{r} \times \vec{X}_{lm} \right) d \quad = 0 \tag{15}$$

Finally

$$\int_{\text{sphere}} \left(\vec{r} \times \vec{X}_{lm} \right) \cdot \left(\vec{r} \times \vec{X}^*_{l'm'} \right) d \quad = \int_{\text{sphere}} \left[r^2 \vec{X}_{lm} \cdot \vec{X}^*_{l'm'} - \left(\vec{r} \cdot \vec{X}_{lm} \right) \left(\vec{r} \cdot \vec{X}^*_{l'm'} \right) \right] d \tag{16}$$

The last term is identically zero, and so:

$$\int_{\text{sphere}} \left(\vec{r}\times \vec{X}_{lm}\right)\cdot\left(\vec{r}\times \vec{X}^*_{l'm'}\right) d \quad = r^2\delta_{ll'}\delta_{mm'} \tag{17}$$

9 Energy and power

The time-averaged energy density in the fields is

$$u = \frac{1}{16\pi}\left(\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right)$$

and so the energy contained in a spherical shell of thickness dr is:

$$\begin{aligned}
dU &= r^2 dr \int u d \\
&= \frac{r^2 dr}{16\pi}\int d \sum_{l,m}\left[a_M(l,m)\, g_l(kr)\,\tilde{\mathbf{X}}_{lm} + \frac{i}{k}a_E(l,m)\,\tilde{\boldsymbol{\nabla}}\times f_l(kr)\,\tilde{\mathbf{X}}_{lm}\right]\cdot \\
&\quad \sum_{l',m'}\left[a_M(l',m')\, g_{l'}(kr)\,\tilde{\mathbf{X}}_{l'm'} + \frac{i}{k}a_E(l',m')\,\tilde{\boldsymbol{\nabla}}\times f_{l'}(kr)\,\tilde{\mathbf{X}}_{l'm'}\right]^* + (B\text{ terms})
\end{aligned}$$

To simplify, note that

$$\begin{aligned}
\tilde{\boldsymbol{\nabla}}\times f_l(kr)\,\tilde{\mathbf{X}}_{lm} &= \frac{\partial f}{\partial r}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm} + f\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{X}}_{lm} \\
&= \frac{\partial f}{\partial r}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm} + \frac{f}{i\sqrt{l(l+1)}}\left[\tilde{\mathbf{r}}\nabla^2 - \tilde{\boldsymbol{\nabla}}\left(1+\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}\right)\right]Y_{lm} \\
&= \frac{\partial f}{\partial r}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm} + \frac{f}{i\sqrt{l(l+1)}}\left(\tilde{\mathbf{r}}\nabla^2 - \tilde{\boldsymbol{\nabla}}\right)Y_{lm}
\end{aligned}$$

In the far zone, $kr \gg 1$, $\frac{\partial f}{\partial r} \sim ikf$ while the terms in gradients of Y_{lm} are of order f/r and are therefore smaller by a factor of $1/kr$. We ignore these terms. Then:

$$\begin{aligned}
dU &= \frac{r^2 dr}{16\pi}\int d \sum_{l,m}\left[a_M(l,m)\, g_l(kr)\,\tilde{\mathbf{X}}_{lm} + \frac{i}{k}a_E(l,m)\frac{\partial}{\partial r}f_l(kr)\,\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm}\right]\cdot \\
&\quad \sum_{l',m'}\left[a_M(l',m')\, g_{l'}(kr)\,\tilde{\mathbf{X}}_{l'm'} + \frac{i}{k}a_E(l',m')\frac{\partial}{\partial r}f_{l'}(kr)\,\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{l'm'}\right]^*
\end{aligned}$$

and we make use of the orthogonality integrals to get:

$$dU = \frac{r^2 dr}{16\pi}\sum_{l,m}\left|a_M(l,m)\, g_l(kr)\right|^2 + \left|\frac{i}{k}a_E(l,m)\frac{\partial}{\partial r}f_l(kr)\right|^2 + (B\text{ terms})$$

Now in the radiation zone,

$$g_l \sim f_l \sim (-i)^{l+1}\frac{e^{ikr}}{kr}$$

and so

$$|g_l| \sim \frac{1}{kr}$$

and

$$\frac{\partial f}{\partial r} \sim ikf$$

Thus the energy is:

$$dU = \frac{dr}{8\pi k^2} \sum_{l,m} |a_M(l,m)|^2 + |a_E(l,m)|^2$$

where the change from the factor 16 to 8 takes into account the equal contribution from the magnetic field.

The total time-averaged power radiated is

$$P = c\frac{dU}{dr} = \frac{c}{8\pi k^2} \sum_{l,m} |a_M(l,m)|^2 + |a_E(l,m)|^2 \tag{18}$$

To find the angular distribution of power radiated, we start with:

$$\begin{aligned} < \frac{dP}{d} >&= \frac{c}{8\pi} r^2 \left|\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^*\right| = \frac{c}{8\pi} r^2 \left|\tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^*\right| \\ &= \frac{c}{8\pi k^2} \left| \begin{array}{c} \sum_{l,m} (-i)^{l+1} \left[a_M(l,m)\,\tilde{\mathbf{X}}_{lm} - a_E(l,m)\,\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{lm}\right] \cdot \\ \sum_{l',m'} i^{l'+1} \left[a_M(l',m')\,\tilde{\mathbf{X}}_{l'm'} - a_E(l',m')\,\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{l'm'}\right]^* \end{array} \right| \end{aligned} \tag{19}$$

where we have used the large argument approximation to the $h^{(1)}$. Note we must include the factor i^{l+1} to be sure we add the different spherical harmonics correctly. The electric and magnetic multipoles for a given l, m have the same angular dependence but different polarizations.

For a single multipole of order (l, m) we have:

$$\begin{aligned} < \frac{dP(l,m)}{d} > &= \frac{c}{8\pi k^2} \left| \left[a_M(l,m)\,\tilde{\mathbf{X}}_{lm} - a_E(l,m)\,\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{lm}\right] \cdot \left[a_M(l,m)\,\tilde{\mathbf{X}}_{lm} - a_E(l,m)\,\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{lm}\right] \right| \\ &= \frac{c}{8\pi k^2} \left(|a_M(l,m)|^2 \left|\tilde{\mathbf{X}}_{lm}\right|^2 + |a_E(l,m)|^2 \left|\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{lm}\right|^2 \right) \\ &= \frac{c}{8\pi k^2} \left(|a_M(l,m)|^2 + |a_E(l,m)|^2 \right) \left|\tilde{\mathbf{X}}_{lm}\right|^2 \end{aligned}$$

where we used the fact that $\left|\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{lm}\right|^2 = \left|\tilde{\mathbf{X}}_{lm}\right|^2$(see equation 16).

We can use equation (14) to write $\left|\tilde{\mathbf{X}}_{lm}\right|^2$ in terms of the angles.

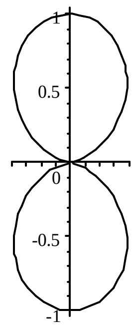
$$\begin{aligned} \left|\tilde{\mathbf{X}}_{lm}\right|^2 &= \frac{1}{l(l+1)} \left(\frac{l(l+1) - m(m+1)}{2} |Y_{l,m+1}|^2 + \frac{l(l+1) - m(m-1)}{2} |Y_{l,m-1}|^2 + m^2 |Y_{lm}|^2 \right) \\ &= \frac{2l+1}{4\pi l(l+1)} \left(\begin{array}{c} \frac{l(l+1)-m(m+1)}{2} \frac{(l-m-1)!}{(l+m+1)!} \left(P_l^{m+1}\right)^2 + \\ \frac{l(l+1)-m(m-1)}{2} \frac{(l-m+1)!}{(l+m-1)!} \left(P_l^{m-1}\right)^2 + m^2 \frac{(l-m)!}{(l+m)!} \left(P_l^m\right)^2 \end{array} \right) \end{aligned}$$

The first few values are:

Dipole: $l = 1$

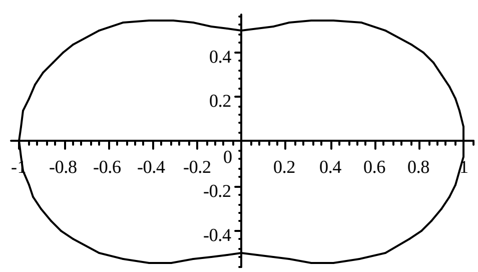
$m = 0$

$$\left|\tilde{\mathbf{X}}_{10}\right|^2 = \frac{3}{8\pi}\left(P_1^1\right)^2 = \frac{3}{8\pi}\sin^2\theta$$



$m = \pm 1$

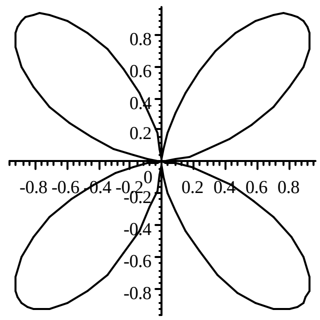
$$\begin{aligned}
\left|\tilde{\mathbf{X}}_{11}\right|^2 &= \frac{3}{8\pi}\left(\frac{2}{2}\left(P_1\right)^2 + \frac{1}{2}\left(P_1^1\right)^2\right)^2 \\
&= \frac{3}{8\pi}\left[\cos^2\theta + \frac{\sin^2\theta}{2}\right] = \frac{3}{8\pi}\left[\cos^2\theta + \frac{1-\cos^2\theta}{2}\right] \\
&= \frac{3}{16\pi}\left(\cos^2\theta + 1\right)
\end{aligned}$$



Quadrupole: $l = 2$

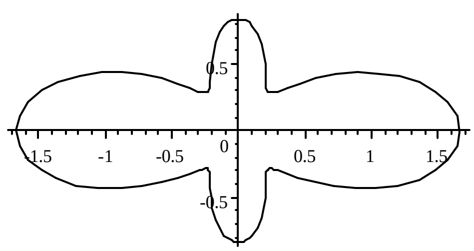
$m = 0$

$$\left|\tilde{\mathbf{X}}_{20}\right|^2 = \frac{5}{24\pi}\frac{6}{3!}\left(P_2^1\right)^2 = \frac{5}{24\pi}\left(-3\sin\theta\cos\theta\right)^2$$
$$= \frac{15}{8\pi}\sin^2\theta\cos^2\theta = \frac{15}{32\pi}\sin^2 2\theta$$



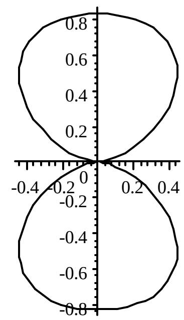
$m = 1$

$$\left|\tilde{\mathbf{X}}_{21}\right|^2 = \frac{1}{6}\left(\frac{6-2}{2}\frac{15}{32\pi}\sin^4\theta + \frac{6}{2}\frac{5}{16\pi}\left(3\cos^2\theta - 1\right)^2 + \frac{15}{8\pi}\sin^2\theta\cos^2\theta\right)$$
$$= \frac{5}{16\pi}\left(4\cos^4\theta - 3\cos^2\theta + 1\right)$$



$m = 2$

$$\left|\tilde{\mathbf{X}}_{22}\right|^2 = \frac{1}{6}\left(\frac{6-2}{2}\frac{15}{8\pi}\sin^2\theta\cos^2\theta + 4\frac{1}{16}\frac{15}{2\pi}\sin^4\theta\right)$$
$$= \frac{5}{16\pi}\left(1 - \cos^4\theta\right)$$



If we have a source with $a_M(l)$ independent of m then

$$
\begin{aligned}
< \frac{dP}{d} >&= \frac{c}{8\pi k^2}\left|\sum_{l,m}(-i)^{l+1}a_M(l)\,\tilde{\mathbf{X}}_{lm}\cdot\sum_{l',m'}i^{l'+1}\left[a_M(l')\,\tilde{\mathbf{X}}_{l'm'}\right]^*\right| \\
&= \frac{c}{8\pi k^2}\left|\sum_{l,l'}(-i)^{l+l'}a_M(l)\,a_M(l')^*\sum_{m,m'}\tilde{\mathbf{X}}_{lm}\cdot\tilde{\mathbf{X}}^*_{l'm'}\right|
\end{aligned}
$$

If the modes superpose incoherently, the expression simplifies:

$$
\begin{aligned}
< \frac{dP}{d} >&= \frac{c}{8\pi k^2}\left|\sum_{l}(-i)^{l+1}a_M(l)\sum_{m=-l}^{+l}\tilde{\mathbf{X}}_{lm}\right|^2 \\
&= \frac{c}{8\pi k^2}\sum_l |a_M(l)|^2\sum_{m=-l}^{+l}\left|\tilde{\mathbf{X}}_{lm}\right|^2 \\
&= \frac{c}{8\pi k^2}\sum_l |a_M(l)|^2\,\frac{2l+1}{4\pi}
\end{aligned}
$$

(cf equation 3.69) and the radiation is isotropic. This is usually the case in atomic systems.

The total power radiated is given by the integral of (19) over solid angle. Because of the orhogonality of the $\tilde{\mathbf{X}}_{lm,}$ the interference terms do not contribute and we regain equation (18).

10 Angular momentum

The angular momentum density is

$$\tilde{\mathbf{m}} = \tilde{\mathbf{r}}\times\tilde{\mathbf{p}} = \frac{1}{4\pi c}\left(\tilde{\mathbf{r}}\times\left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right)\right)$$

and taking the time average, we get

$$\begin{aligned}
< \quad \tilde{\mathbf{m}} >= & \frac{1}{8\pi c}\,\mathrm{Re}\left(\tilde{\mathbf{r}}\times\left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^{*}\right)\right)\\
= & \frac{1}{8\pi c}\,\mathrm{Re}\left(\tilde{\mathbf{E}}\left(\tilde{\mathbf{r}}\cdot\tilde{\mathbf{B}}^{*}\right)-\tilde{\mathbf{B}}^{*}\left(\tilde{\mathbf{r}}\cdot\tilde{\mathbf{E}}\right)\right)\\
= & \frac{1}{8\pi kc}\,\mathrm{Re}\left(\tilde{\mathbf{E}}\left(\tilde{\mathbf{L}}\cdot\tilde{\mathbf{E}}\right)^{*}+\tilde{\mathbf{B}}^{*}\left(\tilde{\mathbf{L}}\cdot\tilde{\mathbf{B}}\right)\right)
\end{aligned}$$

where we used equation (3) and its equivalent for $\tilde{\mathbf{r}}\cdot\tilde{\mathbf{E}}$. For the electric modes the first term is zero and for the magnetic modes the second term is zero.

Let's investigate this expression for the magnetic modes. Using equations (10) and (8), we get

$$< \tilde{\mathbf{m}} >=\frac{1}{8\pi kc}\left(\sum_{l',m'}\frac{\sqrt{l'\left(l'+1\right)}}{k}a_M\left(l',m'\right)g_{l'}\left(kr\right)Y_{l'm'}\right)^{*}\sum_{l,m}\left[a_M\left(l,m\right)g_l\left(kr\right)\tilde{\mathbf{X}}_{lm}\right]$$

and the total angular momentum in a spherical shell between r and $r+dr$ is:

$$d\tilde{\mathbf{M}}=\frac{r^2dr}{8\pi kc}\,\mathrm{Re}\sum_{l',m'}\sum_{l,m}\frac{\sqrt{l'\left(l'+1\right)}}{k\sqrt{l\left(l+1\right)}}a_M^{*}\left(l',m'\right)g_{l'}^{*}\left(kr\right)g_l a_M\left(l,m\right)\int\left(\tilde{\mathbf{L}}Y_{lm}\right)^{*}Y_{l'm'}d$$

Now we use the expression we derived for $\tilde{\mathbf{L}}Y_{lm}$ in equation (14)

$$\int\left(\tilde{\mathbf{L}}Y_{lm}\right)^{*}Y_{l'm'}d \quad =-\int\left(\begin{array}{c}\sqrt{l\left(l+1\right)-m\left(m+1\right)}Y_{l,m+1}\left(\frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{2}\right)+\\ \sqrt{l\left(l+1\right)-m\left(m-1\right)}Y_{l,m-1}\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)+mY_{lm}\hat{\mathbf{z}}\end{array}\right)^{*}Y_{l'm'}d$$

Thus we get contributions only for $l=l'$ and $m'=m-1$, m, or $m+1$.

$$\int\left(\tilde{\mathbf{L}}Y_{lm}\right)^{*}Y_{l'm'}d \quad =\delta_{ll'}\left[\begin{array}{c}\sqrt{l\left(l+1\right)-m\left(m+1\right)}\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)\delta_{m+1,m'}+\\ \sqrt{l\left(l+1\right)-m\left(m-1\right)}\left(\frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{2}\right)\delta_{m-1,m'}+m\delta_{mm'}\end{array}\right]$$

and thus

$$d\tilde{\mathbf{M}}=\frac{r^2dr}{8\pi kc}\,\mathrm{Re}\sum_{l,m}a_M\left(l,m\right)^{*}a_M\left(l,m'\right)\left|f_l\left(kr\right)\right|^2\left[\begin{array}{c}\sqrt{l\left(l+1\right)-m\left(m+1\right)}\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)\delta_{m+1,m'}+\\ \sqrt{l\left(l+1\right)-m\left(m-1\right)}\left(\frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{2}\right)\delta_{m-1,m'}+m\delta_{mm'}\end{array}\right]$$

In the radiation zone, $\left|f_l\left(kr\right)\right|^2=\frac{1}{\left(kr\right)^2}$

$$dM_x=\frac{dr}{16\pi k^3c}\,\mathrm{Re}\sum_{l,m}a_M\left(l,m\right)^{*}\left(\begin{array}{c}\sqrt{l\left(l+1\right)-m\left(m+1\right)}a_M\left(l,m+1\right)\\ +\sqrt{l\left(l+1\right)-m\left(m-1\right)}a_M\left(l,m-1\right)\end{array}\right)$$

$$dM_y=\frac{dr}{16\pi k^3c}\,\mathrm{Im}\sum_{l,m}a_M\left(l,m\right)^{*}\left(\begin{array}{c}\sqrt{l\left(l+1\right)-m\left(m+1\right)}a_M\left(l,m+1\right)\\ -\sqrt{l\left(l+1\right)-m\left(m-1\right)}a_M\left(l,m-1\right)\end{array}\right)$$

$$dM_z=\frac{dr}{8\pi k^3c}\sum_{l,m}m\left|a_M\left(l,m\right)\right|^2$$

and comparing with equation (18), we find that for a single multipole

$$\frac{dM_z}{dr} = \frac{m}{\omega}\frac{dU}{dr}$$

suggesting that the radiation from an (l, m) multipole carries off $m\hbar$ units of $z-$component of angular momentum per photon of energy $\hbar\omega$. This is consistent with the quantum mechanical interpretation. However the x and $y-$components are more complicated. See Jackson for a more extensive discussion.

11 Connection with sources

We have already noted that the coefficients a_E and a_M are determined by the radial components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ evaluated on some spherical surface. Now we want to relate these quantities to the properties of the sources - the charges and currents. (See Jackson for the additional sources of magnetization.)

As usual we Fourier transform all physical quantities:

$$\rho\left(\tilde{\mathbf{x}}, \mathbf{t}\right) = \frac{1}{\sqrt{2\pi}} \int \rho\left(\tilde{\mathbf{x}}, \omega\right) e^{-i\omega t} d\omega$$

Now we define a new field-like quantity

$$\tilde{\mathbf{F}} = \tilde{\mathbf{E}} + \frac{4\pi i}{\omega}\tilde{\mathbf{J}}$$

so that

$$\tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{F}} = \tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{E}} + \frac{4\pi i}{\omega}\tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{J}} = \tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{E}} + \frac{4\pi i}{\omega}\left(i\omega\rho\right) = \tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{E}} - 4\pi\rho = 0$$

where we used charge continuity to replace $\tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{J}}$ with $i\omega\rho$. Outside the source, $\tilde{\mathbf{F}} = \tilde{\mathbf{E}}$. Now

$$\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{F}} = \tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{E}} + \frac{4\pi i}{\omega}\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{J}} = ik\tilde{\mathbf{B}} + \frac{4\pi i}{\omega}\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{J}}$$

and

$$\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{B}} = \frac{4\pi}{c}\tilde{\mathbf{J}} - i\frac{\omega}{c}\tilde{\mathbf{E}} = -ik\left(\tilde{\mathbf{E}} + \frac{4\pi i}{\omega}\tilde{\mathbf{J}}\right) = -ik\tilde{\mathbf{F}}$$

Now we find the wave equation for $\tilde{\mathbf{F}}$ in the usual way:

$$\begin{aligned} \tilde{\boldsymbol{\nabla}} \times \left(\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{F}}\right) &= \tilde{\boldsymbol{\nabla}} \times \left(ik\tilde{\mathbf{B}} + \frac{4\pi i}{\omega}\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{J}}\right) \\ \tilde{\boldsymbol{\nabla}}\left(\tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{F}}\right) - \nabla^2\tilde{\mathbf{F}} &= ik\left(-ik\tilde{\mathbf{F}}\right) + \frac{4\pi i}{\omega}\tilde{\boldsymbol{\nabla}} \times \left(\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{J}}\right) \end{aligned}$$

So we have

$$\left(\nabla^2 + k^2\right)\tilde{\mathbf{F}} = -\frac{4\pi i}{\omega}\tilde{\boldsymbol{\nabla}} \times \left(\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{J}}\right)$$

and similarly

$$
\begin{aligned}
\tilde{\boldsymbol{\nabla}}\times\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{B}}\right) &= -ik\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{F}} \\
\tilde{\boldsymbol{\nabla}}\left(\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{B}}\right)-\nabla^2\tilde{\mathbf{B}} &= -ik\left(ik\tilde{\mathbf{B}}+\frac{4\pi i}{\omega}\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right) \\
\left(\nabla^2+k^2\right)\tilde{\mathbf{B}} &= -\frac{4\pi}{c}\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}
\end{aligned}
$$

Now we convert these equations to equations in $\tilde{\mathbf{r}}\cdot\tilde{\mathbf{F}}$ and $\tilde{\mathbf{r}}\cdot\tilde{\mathbf{B}}$ as we did in section 3.

$$
\begin{aligned}
\left(\nabla^2+k^2\right)\tilde{\mathbf{r}}\cdot\tilde{\mathbf{F}} &= -\frac{4\pi i}{\omega}\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}\times\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right) \\
&= \frac{4\pi}{\omega}\tilde{\mathbf{L}}\cdot\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right)
\end{aligned}
$$

and

$$
\left(\nabla^2+k^2\right)\tilde{\mathbf{r}}\cdot\tilde{\mathbf{B}} = -\frac{4\pi}{c}\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}} = -\frac{4\pi i}{c}\tilde{\mathbf{L}}\cdot\tilde{\mathbf{J}}
$$

(cf equation 3). Now we can use the Green's function that we found previously (eqn. 2):

$$
\begin{aligned}
\tilde{\mathbf{r}}\cdot\tilde{\mathbf{F}} &= \frac{-1}{\omega}\int\frac{e^{ikR}}{R}\tilde{\mathbf{L}}'\cdot\left(\tilde{\boldsymbol{\nabla}}'\times\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)\right)d^3\tilde{\mathbf{x}}' \\
&= \frac{-ik}{\omega}4\pi\int\sum_{l,m}j_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right)Y_{lm}\left(\theta,\phi\right)Y_{lm}^*\left(\theta',\phi'\right)\tilde{\mathbf{L}}'\cdot\left(\tilde{\boldsymbol{\nabla}}'\times\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)\right)d^3\tilde{\mathbf{x}}'
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\mathbf{r}}\cdot\tilde{\mathbf{B}} &= \frac{i}{c}\int\frac{e^{ikR}}{R}\tilde{\mathbf{L}}'\cdot\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)d^3\tilde{\mathbf{x}}' \\
&= \frac{-k}{c}4\pi\int\sum_{l,m}j_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right)Y_{lm}\left(\theta,\phi\right)Y_{lm}^*\left(\theta',\phi'\right)\tilde{\mathbf{L}}'\cdot\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)d^3\tilde{\mathbf{x}}'
\end{aligned}
$$

then equation 11 for the coefficient becomes:

$$
\begin{aligned}
a_M\left(l',m'\right)g_{l'}\left(kr\right) &= \frac{k4\pi}{\sqrt{l'\left(l'+1\right)}}\int\frac{-k}{c}\int\sum_{l,m}j_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right)Y_{lm}\left(\theta,\phi\right)Y_{lm}^*\left(\theta',\phi'\right)\tilde{\mathbf{L}}'\cdot\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)d^3\tilde{\mathbf{x}}'Y_{l'm'}^*\,(\theta \\
a_M\left(l,m\right)g_l\left(kr\right) &= \frac{-4\pi k^2}{c\sqrt{l\left(l+1\right)}}h_l^{(1)}\left(kr\right)\int j_l\left(kr'\right)Y_{lm}^*\left(\theta',\phi'\right)\tilde{\mathbf{L}}'\cdot\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)d^3\tilde{\mathbf{x}}'
\end{aligned}
$$

where in the last step we took $r_> = r$ outside the source.

Similarly for a_E we use the fact that $\tilde{\mathbf{F}}\equiv\tilde{\mathbf{E}}$ outside the source, so:

$$
\begin{aligned}
a_E\left(l',m'\right)f_{l'}\left(kr\right) &= \frac{-k4\pi}{\sqrt{l'\left(l'+1\right)}}\int\frac{-i}{c}\int\sum_{l,m}j_l\left(kr_<\right)h_l^{(1)}\left(kr_>\right)Y_{lm}\left(\theta,\phi\right)Y_{lm}^*\left(\theta',\phi'\right)\tilde{\mathbf{L}}'\cdot\left(\tilde{\boldsymbol{\nabla}}'\times\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)\right)d^3\tilde{\mathbf{x}}' \\
a_E\left(l,m\right)g_l\left(kr\right) &= \frac{ik4\pi}{c\sqrt{l\left(l+1\right)}}h_l^{(1)}\left(kr\right)\int j_l\left(kr'\right)Y_{lm}^*\left(\theta',\phi'\right)\tilde{\mathbf{L}}'\cdot\left(\tilde{\boldsymbol{\nabla}}'\times\tilde{\mathbf{J}}\left(\tilde{\mathbf{x}}'\right)\right)d^3\tilde{\mathbf{x}}'
\end{aligned}
$$

So far these expressions are exact.

Now we'd like to express the results in terms of the multipole moments of the source.

First we evaluate:

$$\begin{aligned}\tilde{\mathbf{L}}\cdot\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right) &= \frac{1}{i}\left(\tilde{\mathbf{r}}\times\tilde{\boldsymbol{\nabla}}\right)\cdot\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right)\\ &= \frac{1}{i}\left(r_j\partial_k\partial_j J_k - r_j\nabla^2 J_j\right)\end{aligned}$$

where we used a result from the front cover of Jackson. Now we work on the last term:

$$\begin{aligned}r_j\nabla^2 J_j &= \partial_k\left(r_j\partial_k J_j\right)-\left(\partial_k r_j\right)\partial_k J_j\\ &= \partial_k\left(\partial_k r_j J_j - \left(\partial_k r_j\right)J_j\right)-\delta_{kj}\partial_k J_j\\ &= \nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)-\partial_k\delta_{kj}J_j-\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}\\ &= \nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)-2\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}\end{aligned}$$

and thus

$$\begin{aligned}\tilde{\mathbf{L}}\cdot\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}\right) &= \frac{1}{i}\left[\left(\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}-\nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)+2\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}\right]\\ &= \frac{1}{i}\left[\left(2+r\frac{\partial}{\partial r}\right)\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}-\nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)\right]\\ &= i\left[\frac{-1}{r}\frac{\partial}{\partial r}\left(r^2\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{J}}\right)+\nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)\right]\\ &= i\left[\frac{-1}{r}\frac{\partial}{\partial r}\left(i\omega\rho r^2\right)+\nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)\right]\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(\omega\rho r^2\right)+i\nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)\end{aligned}$$

and so equation (21) with $g_l = h_l^{(1)}$ becomes:

$$a_E\left(l,m\right)=\frac{ik4\pi}{c\sqrt{l\left(l+1\right)}}\int j_l\left(kr\right)Y_{lm}^*\left(\theta,\phi\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\omega\rho r^2\right)+i\nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)\right)d^3\tilde{\mathbf{x}}$$

Now the first term may be rewritten using integration by parts in r :

$$\int j_l\left(kr\right)\frac{1}{r}\frac{\partial}{\partial r}\left(\omega\rho r^2\right)r^2 dr = \left. rj_l\left(kr\right)\omega\rho r^2\right|_0^{r_{\max}}-\int\frac{\partial}{\partial r}\left(rj_l\left(kr\right)\right)\omega\rho r^2 dr$$

where the integrated term is zero provided we take $r_{\max}$ outside the source. Then:

$$a_E\left(l,m\right)=\frac{k^2 4\pi}{\sqrt{l\left(l+1\right)}}\int Y_{lm}^*\left(\theta,\phi\right)\left(-i\rho\frac{\partial}{\partial r}\left(rj_l\left(kr\right)\right)-\frac{j_l\left(kr\right)}{kc}\nabla^2\left(\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}\right)\right)d^3\tilde{\mathbf{x}}$$

Now we work on the last term. From Green's theorem

$$\int\left(\psi\nabla^2\phi-\phi\nabla^2\psi\right)dV=\int\left(\psi\frac{\partial\phi}{\partial n}-\phi\frac{\partial\psi}{\partial n}\right)dA$$

Now we take $\phi=\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}$ and $\psi=j_l\left(kr\right)Y_{lm}^*\left(\theta,\phi\right)$. Then $\phi=0$ on the surface provided it is outside the source, so the right hand side is zero. Also,

$$\nabla^2 j_l\left(kr\right)Y_{lm}^*=-k^2 j_l\left(kr\right)Y_{lm}^*$$

and so finally we have:

$$a_E(l,m)=\frac{k^2 4\pi}{\sqrt{l(l+1)}}\int Y^*_{lm}(\theta,\phi)\left(-i\rho\frac{\partial}{\partial r}(rj_l(kr))+kj_l(kr)\left(\frac{\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}}{c}\right)\right)d^3\tilde{\mathbf{x}} \quad (22)$$

Now what about a_M : We just need to rearrange

$$\begin{aligned}\tilde{\mathbf{L}}\cdot\tilde{\mathbf{J}} &= \frac{1}{i}\left(\tilde{\mathbf{r}}\times\tilde{\boldsymbol{\nabla}}\right)\cdot\tilde{\mathbf{J}}=\frac{1}{i}\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{J}}=\frac{1}{i}\left(\tilde{\mathbf{J}}\cdot\left(\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{r}}\right)-\tilde{\boldsymbol{\nabla}}\cdot\left(\tilde{\mathbf{r}}\times\tilde{\mathbf{J}}\right)\right)\\ &= \frac{-1}{i}\tilde{\boldsymbol{\nabla}}\cdot\left(\tilde{\mathbf{r}}\times\tilde{\mathbf{J}}\right)=i\tilde{\boldsymbol{\nabla}}\cdot\left(\tilde{\mathbf{r}}\times\tilde{\mathbf{J}}\right)\end{aligned}$$

$$a_M(l,m)=\frac{-4\pi ik^2}{c\sqrt{l(l+1)}}\int j_l(kr)\,Y^*_{lm}(\theta,\phi)\,\tilde{\boldsymbol{\nabla}}\cdot\left(\tilde{\mathbf{r}}\times\tilde{\mathbf{J}}\right)d^3\tilde{\mathbf{x}} \quad (23)$$

These expressions are still exact.

Example: the center-fed linear antenna.

Suppose an antenna of length d has a current distribution

$$I=I_0\sin\left[k\left(\frac{d}{2}-r\right)\right]e^{-i\omega t}$$

or equivalently,

$$\vec{j}=\frac{I_0}{2\pi r^2}\sin\left(k\left(\frac{d}{2}-r\right)\right)\left[\delta(\mu-1)-\delta(\mu+1)\right]\hat{\mathbf{r}}$$

Then from the charge conservation, we get:

$$\rho=\frac{1}{i\omega}\vec{\nabla}\cdot\vec{j}=\frac{-kI_0}{2\pi i\omega r^2}\cos\left(k\left(\frac{d}{2}-r\right)\right)\left[\delta(\mu-1)-\delta(\mu+1)\right]$$

but since $\omega=kc$, then

$$\rho=\frac{iI_0}{2\pi cr^2}\cos\left(k\left(\frac{d}{2}-r\right)\right)\left[\delta(\mu-1)-\delta(\mu+1)\right]$$

Now for this antenna, $\vec{r}\times\vec{j}=0$, so $a_M=0$. Then from equation (22):

$$\begin{aligned}a_E(l,m) &= \frac{k^2 2I_0}{c\sqrt{l(l+1)}}\int Y^*_{lm}(\theta,\phi)\frac{\left[\delta(\mu-1)-\delta(\mu+1)\right]}{r^2}\left(\begin{array}{c}\cos\left(k\left(\frac{d}{2}-r\right)\right)\frac{\partial}{\partial r}(rj_l(kr))\\ +krj_l(kr)\sin\left(k\left(r-\frac{d}{2}\right)\right)\end{array}\right)d^3\vec{x}\\ &= \frac{4\pi k^2 I_0}{c\sqrt{l(l+1)}}\int\left[Y^*_{l0}(0)-Y^*_{l0}(\pi)\right]\left(\begin{array}{c}\cos\left(k\left(\frac{d}{2}-r\right)\right)\frac{\partial}{\partial r}(rj_l(kr))\\ +krj_l(kr)\sin\left(k\left(r-\frac{d}{2}\right)\right)\end{array}\right)dr\end{aligned}$$

where

$$\begin{aligned}Y^*_{l0}(0)-Y^*_{l0}(\pi) &= \sqrt{\frac{2l+1}{4\pi}}\left(P_l(1)-P_l(-1)\right)\\ &= \sqrt{\frac{2l+1}{4\pi}}\left(1-(-1)^l\right)=\begin{cases}0 & \text{if } \ l \text{ is even}\\ \sqrt{\frac{2l+1}{\pi}} & \text{if } \ l \text{ is odd}\end{cases}\end{aligned}$$

Next we rearrange the first term in the integrand:

$$\cos\left(k\left(\frac{d}{2}-r\right)\right)\frac{\partial}{\partial r}\left(rj_l\left(kr\right)\right)=\frac{\partial}{\partial r}\left[\cos\left(k\left(\frac{d}{2}-r\right)\right)rj_l\left(kr\right)\right]-k\sin\left(k\left(\frac{d}{2}-r\right)\right)rj_l\left(kr\right)$$

Thus

$$\begin{aligned}
a_E\left(l,0\right) &= \frac{4k^2I_0\sqrt{\left(2l+1\right)\pi}}{c\sqrt{l\left(l+1\right)}}\int\frac{\partial}{\partial r}\left[\cos\left(k\left(\frac{d}{2}-r\right)\right)rj_l\left(kr\right)\right]dr \\
&= \frac{4k^2I_0\sqrt{\left(2l+1\right)\pi}}{c\sqrt{l\left(l+1\right)}}\cos\left(k\left(\frac{d}{2}-r\right)\right)rj_l\left(kr\right)\Big|_0^{d/2} \\
&= \frac{4k^2I_0\sqrt{\left(2l+1\right)\pi}}{c\sqrt{l\left(l+1\right)}}\frac{d}{2}j_l\left(\frac{kd}{2}\right) \\
&= \frac{8I_0}{cd}\sqrt{\frac{\left(2l+1\right)\pi}{l\left(l+1\right)}}\left(\frac{kd}{2}\right)^2j_l\left(\frac{kd}{2}\right)
\end{aligned}$$

where l is odd. Thus there are multipoles of all odd orders.

The full wave ($kd=2\pi$) and half wave ($kd=\pi$) are of particular interest. For these we find the coefficients $cda_E\left(l,0\right)/8\sqrt{\pi}I_0$ to be:

l	full wave	half wave
1	$\frac{1}{2}\sqrt{6}\pi^2j_1\left(\pi\right)=\frac{1}{2}\sqrt{6}\pi^2\left(\frac{\sin\pi}{\pi^2}-\frac{\cos\pi}{\pi}\right)$ $=\frac{\pi}{2}\sqrt{6}=3.848$	$\frac{1}{2}\sqrt{6}\frac{\pi}{4}^2j_1\left(\frac{\pi}{2}\right)=\frac{1}{2}\sqrt{6}\frac{\pi}{4}^2\left(\frac{4}{\pi^2}\right)=\frac{\sqrt{6}}{8}=0.305$
3	$\frac{1}{6}\sqrt{21}\pi^2j_3\left(\pi\right)=1.2473$	$\frac{1}{24}\sqrt{21}\pi^2j_l\left(\frac{\pi}{2}\right)=6.0544\times10^{-2}$
5	$\frac{1}{30}\sqrt{330}\pi^2j_5\left(\pi\right)=0.11914$	$\frac{1}{480}\sqrt{330}\pi^2j_5\left(\frac{\pi}{2}\right)=3.1233\times10^{-4}$
7	$\frac{1}{28}\sqrt{210}\pi^2j_7\left(\pi\right)=5.6673\times10^{-3}$	$\frac{1}{112}\sqrt{210}\pi^2j_7\left(\frac{\pi}{2}\right)=2.0037\times10^{-3}$

Finally we need the $\vec{X}_{l0}$ (equation 14)

$$\begin{aligned}
\vec{X}_{l0} &= \frac{-1}{\sqrt{l\left(l+1\right)}}\left(\frac{1}{2}\sqrt{l\left(l+1\right)}Y_{l,1}\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right)+\frac{1}{2}\sqrt{l\left(l+1\right)}Y_{l,-1}\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)\right) \\
&= \frac{-1}{2}\sqrt{\frac{2l+1}{4\pi}}\left(\sqrt{\frac{\left(l-1\right)!}{\left(l+1\right)!}}P_l^1e^{i\phi}\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right)+\sqrt{\frac{\left(l+1\right)!}{\left(l-1\right)!}}P_l^{-1}e^{-i\phi}\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)\right) \\
&= \frac{-1}{2}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{\left(l-1\right)!}{\left(l+1\right)!}}\left(-1\right)\sin\theta\frac{d}{d\mu}P_l\left(\mu\right)\left(e^{i\phi}\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right)-e^{-i\phi}\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)\right) \\
&= i\sqrt{\frac{2l+1}{4\pi}\frac{\left(l-1\right)!}{\left(l+1\right)!}}\sin\theta\frac{d}{d\mu}P_l\left(\mu\right)\left(\sin\phi\hat{\mathbf{x}}-\cos\phi\hat{\mathbf{y}}\right) \\
&= -i\sqrt{\frac{2l+1}{4\pi}\frac{\left(l-1\right)!}{\left(l+1\right)!}}\sin\theta\frac{d}{d\mu}P_l\left(\mu\right)\hat{\boldsymbol{\phi}}
\end{aligned}$$

and thus

$$\hat{\mathbf{r}}\times\vec{X}_{l0}=i\sqrt{\frac{2l+1}{4\pi}\frac{\left(l-1\right)!}{\left(l+1\right)!}}\sin\theta\frac{d}{d\mu}P_l\left(\mu\right)\hat{\boldsymbol{\theta}}$$

and

$$
\begin{aligned}
a_E(l,0)\,\hat{\mathbf{r}}\times\vec{X}_{l0} &= \frac{8I_0}{cd}\sqrt{\frac{(2l+1)\pi}{l(l+1)}}\left(\frac{kd}{2}\right)^2 j_l\left(\frac{kd}{2}\right) i\sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}}\sin\theta\frac{d}{d\mu}P_l(\mu)\,\hat{\boldsymbol{\theta}} \\
&= i\frac{4I_0}{cd}\frac{(2l+1)}{l(l+1)}\left(\frac{kd}{2}\right)^2 j_l\left(\frac{kd}{2}\right)\sin\theta\frac{d}{d\mu}P_l(\mu)\,\hat{\boldsymbol{\theta}}
\end{aligned}
$$

Thus the power is:

$$
\begin{aligned}
&< \frac{dP}{d} >= \frac{c}{8\pi k^2}\left|\sum_{l\text{ odd}}(-i)^{l+1}\left[-a_E(l)\,\hat{\mathbf{r}}\times\vec{X}_{l0}\right]\cdot\sum_{l'} i^{l'+1}\left[-a_E(l')\,\hat{\mathbf{r}}\times\vec{X}_{l'0}\right]^*\right| \\
&= \frac{c}{8\pi k^2}\left|\sum_{l\text{ odd}}(-i)^l\, i\frac{4I_0}{cd}\frac{(2l+1)}{l(l+1)}\left(\frac{kd}{2}\right)^2 j_l\left(\frac{kd}{2}\right)\sin\theta\frac{d}{d\mu}P_l(\mu)\cdot\sum_{l'}\left(\begin{array}{c}\text{same with } l\to l' \\ \text{and } i\to -i\end{array}\right)\right| \\
&= \frac{c}{8\pi k^2}\left(\frac{4I_0}{cd}\right)^2\left(\frac{kd}{2}\right)^4\sin^2\theta\left|\sum_l(-1)^{\frac{l-1}{2}}\frac{(2l+1)}{l(l+1)}j_l\left(\frac{kd}{2}\right)\frac{d}{d\mu}P_l(\mu)\left(\begin{array}{c}\text{same} \\ \text{with } l\to l'\end{array}\right)\right| \\
&= \frac{I_0^2}{2\pi c}\left(\frac{kd}{2}\right)^2\sin^2\theta\left|\sum_l(-1)^{\frac{l-1}{2}}\frac{2l+1}{l(l+1)}j_l\left(\frac{kd}{2}\right)\frac{d}{d\mu}P_l(\mu)\,(\text{same with } l\to l')\right|
\end{aligned}
$$

Thus for the full wave antenna, we get:

$$
\begin{aligned}
&< \frac{dP}{d} >= \frac{I_0^2}{2\pi c}\pi^2\sin^2\theta\left|\left(\begin{array}{c}\frac{3}{2}j_1(\pi)\frac{d}{d\mu}(\mu)-\frac{7}{12}j_3(\pi)\frac{d}{d\mu}\left(\frac{1}{2}\left(5\mu^3-3\mu\right)\right) \\ +\frac{11}{30}j_5(\pi)\frac{d}{d\mu}\left(\frac{63}{8}\mu^5-\frac{35}{4}\mu^3+\frac{15}{8}\mu\right)\end{array}\right)(\text{same})\right| \\
&= \frac{I_0^2}{2\pi c}\pi^2\sin^2\theta\left(\frac{3}{2}j_1(\pi)-\frac{7}{8}j_3(\pi)\left(5\mu^2-1\right)+\frac{11}{8}j_5(\pi)\left(\frac{21}{2}\mu^4-7\mu^2+\frac{1}{2}\right)\right)^2 \\
&= \frac{I_0^2}{2\pi c}\pi^2\sin^2\theta\left(\frac{3}{2}j_1(\pi)\right)^2\left(1-\frac{7}{12}\frac{j_3(\pi)}{j_1(\pi)}\left(5\mu^2-1\right)+\frac{11}{12}\frac{j_5(\pi)}{j_1(\pi)}\left(\frac{21}{2}\mu^4-7\mu^2+\frac{1}{2}\right)\right)^2
\end{aligned}
$$

But

$$
j_1(\pi)=\frac{\sin\pi}{\pi^2}-\frac{\cos\pi}{\pi}=\frac{1}{\pi}=0.318\,31
$$

so

$$
\begin{aligned}
&< \frac{dP}{d} >= \frac{I_0^2}{2\pi c}\frac{9}{4}\sin^2\theta\left(1-\frac{7}{12}\frac{J_{3.5}(\pi)}{J_{1.5}(\pi)}\left(5\mu^2-1\right)+\frac{11}{12}\frac{J_{5.5}(\pi)}{J_{1.5}(\pi)}\left(21\mu^4-14\mu^2+1\right)\right)^2 \\
&= \frac{I_0^2}{2\pi c}\frac{9}{4}\sin^2\theta\left(1-0.303\,23\left(5\mu^2-1\right)+5.741\,0\times10^{-2}\left(21\mu^4-14\mu^2+1\right)\right)^2 \\
&= \frac{9}{4}\frac{I_0^2}{2\pi c}\sin^2\theta\left(1.360\,6-2.319\,9\mu^2+1.205\,6\mu^4\right)^2 \\
&= \frac{9}{4}\frac{I_0^2}{2\pi c}\sin^2\theta\left(1.8125-5.7054\mu^2+6.9246\mu^4-3.8319\mu^6+0.81757\mu^8\right)
\end{aligned}
$$

Jackson's exact result (9.57) is the blue-green line

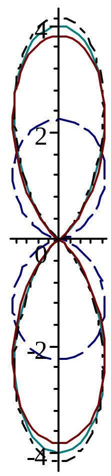
$$
< \frac{dP}{d} >= \frac{I^2}{2\pi c}\frac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}
$$

Two term result (red line) :

$\frac{9}{4}(\sin\theta)^2\left(1-0.30323\left(5(\cos\theta)^2-1\right)\right)^2$

the 3 term result is almost the same (black line)

$\frac{9}{4}(\sin\theta)^2\left(1.8125-5.7054\cos^2\theta+6.9246\cos^4\theta-3.8319\cos^6\theta+.81757\cos^8\theta\right)$



The blue dashed line is the 1-term (dipole) result.

Half-wave antenna:

$$<\frac{dP}{d}>=\frac{I_0^2}{2\pi c}\frac{\pi^2}{4}\sin^2\theta\left|\left(\begin{array}{c}\frac{3}{2}j_1\left(\frac{\pi}{2}\right)\frac{d}{d\mu}(\mu)-\frac{7}{12}j_3\left(\frac{\pi}{2}\right)\frac{d}{d\mu}\left(\frac{1}{2}\left(5\mu^3-3\mu\right)\right)\\+\frac{11}{30}j_5\left(\frac{\pi}{2}\right)\frac{d}{d\mu}\left(\frac{63}{8}\mu^5-\frac{35}{4}\mu^3+\frac{15}{8}\mu\right)\end{array}\right)(\text{same})\right|$$

Here

$$j_1\left(\frac{\pi}{2}\right)=\frac{4}{\pi^2}$$

so

$$<\frac{dP}{d}>=\frac{I_0^2}{2\pi c}\frac{9}{4}\frac{\pi^2}{4}\left(\frac{4}{\pi^2}\right)^2\sin^2\theta\left(1-\frac{7}{12}\frac{J_{3.5}\left(\frac{\pi}{2}\right)}{J_{1.5}\left(\frac{\pi}{2}\right)}\left(5\mu^2-1\right)+\frac{11}{12}\frac{J_{5.5}\left(\frac{\pi}{2}\right)}{J_{1.5}\left(\frac{\pi}{2}\right)}\left(21\mu^4-14\mu^2+1\right)\right)^2$$

$$=\frac{I_0^2}{2\pi c}\frac{9}{\pi^2}\sin^2\theta\left(1-4.6241\times10^{-2}\left(5\mu^2-1\right)+1.8912\times10^{-3}\left(21\mu^4-14\mu^2+1\right)\right)^2$$

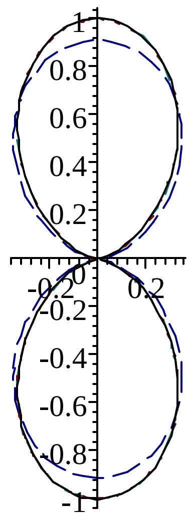
$$=\frac{I_0^2}{2\pi c}\frac{9}{\pi^2}\sin^2\theta\left(1.0481-0.25768\mu^2+3.9715\times10^{-2}\mu^4\right)^2$$

$(\sin\theta)^2\frac{9}{\pi^2}\left(1.0481-0.25768\cos^2\theta+3.9715\times10^{-2}\cos^4\theta\right)^2$

$(\sin\theta)^2\frac{9}{\pi^2}\left(1-4.6241\times10^{-2}\left(5\cos^2\theta-1\right)\right)^2$

Jackson's exact result (9.57) is the blue-green line

$$< \frac{dP}{d} >= \frac{I^2}{2\pi c} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$



Here the two-term (red dotted) and 3-term(black) results are indistinguishable. Again the dashed blue

line is the 1-term (dipole) result.

Now we specialize to the case in which the source size $d \ll \lambda$. This means that $kr \ll 1$ in each integrand, and we may use the small argument expansion for the Bessel function j_l :

$$j_l(kr) = \frac{(kr)^l}{(2l+1)!!}$$

and

$$\frac{\partial}{\partial r}(r j_l(kr)) = \frac{\partial}{\partial r} \frac{k^l r^{l+1}}{(2l+1)!!} = \frac{l+1}{(2l+1)!!}(kr)^l$$

The quantity that appears in the integrand of a_E (equation 22) is

$$\begin{aligned} -i\rho\frac{\partial}{\partial r}(r j_l(kr)) + k j_l(kr)\left(\frac{\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}}}{c}\right) &= -i\rho\frac{l+1}{(2l+1)!!}(kr)^l + \frac{k}{c}\frac{(kr)^l}{(2l+1)!!}\tilde{\mathbf{J}}\cdot\tilde{\mathbf{r}} \\ &= \frac{(kr)^l}{(2l+1)!!}\left(-i\rho(l+1) + kr\frac{\tilde{\mathbf{J}}\cdot\hat{\mathbf{r}}}{c}\right) \end{aligned}$$

The second term is smaller than the first by a factor kr, and so we drop it. Then:

$$\begin{aligned} a_E(l,m) &= \frac{-ik^2 4\pi}{\sqrt{l(l+1)}} \frac{l+1}{(2l+1)!!} \int Y_{lm}^*(\theta,\phi)\,\rho\,(kr)^l\,d^3\tilde{\mathbf{x}} \\ &= \frac{-4\pi i k^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}Q_{lm} \end{aligned}$$

where

$$Q_{lm} = \int \rho r^l Y_{lm}^*(\theta,\phi)\,d^3\tilde{\mathbf{x}}$$

is the spherical multipole moment of the source.

Similarly,

$$a_M(l,m) = \frac{4\pi i k^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}M_{lm}$$

where

$$M_{lm} = \frac{-1}{l+1}\int \tilde{\boldsymbol{\nabla}}\cdot\left(\tilde{\mathbf{r}}\times\frac{\tilde{\mathbf{J}}}{c}\right) r^l Y_{lm}^*(\theta,\phi)$$

is the magnetic multipole.

12 Atomic systems

The transition probability is defined in terms of the radiated power by

$$\frac{P}{\hbar\omega} = \frac{1}{\tau} = \text{ transition probability}$$

where τ is the mean lifetime of the state. The charge density is assumed to have the form

$$\rho(\tilde{\mathbf{x}}) = \begin{cases} \frac{3e}{a^3}Y_{lm}(\theta,\phi) & for \quad r<a \\ 0 & for \quad r>a \end{cases}$$

So that

$$\begin{aligned} Q_{l'm'} &= \frac{3e}{a^3}\int_0^a r^l r^2 dr \int Y_{lm}(\theta,\phi)\,Y_{l'm'}^*(\theta,\phi)\,d \\ &= \frac{3e}{a^3}\frac{a^{l+3}}{l+3}\delta_{ll'}\delta_{mm'} \\ &= \frac{3e}{l+3}a^l\delta_{ll'}\delta_{mm'} \end{aligned}$$

and thus

$$a_E(l,m) = \frac{-4\pi i k^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\frac{3e}{l+3}a^l$$

Then from equation (18) for the power, we have

$$\begin{aligned}\frac{1}{\tau} &\simeq \frac{c}{8\pi k^2\hbar\omega}\left(\frac{3e}{l+3}a^l\right)^2\frac{(4\pi)^2 k^{2l+4}}{[(2l+1)!!]^2}\frac{l+1}{l} \\ &= \frac{e^2}{\hbar c}\left(\frac{3}{l+3}\right)^2\left(\frac{l+1}{l}\right)\frac{2\pi}{[(2l+1)!!]^2}(ka)^{2l}\,\omega\end{aligned}$$

Because of the factor $(ka)^{2l}$, the transition probability falls off rapidly with increasing l,and so the lowest non-vanishing multipole is usually the only one we have to consider. The magnetic multipole contribution is smaller by a factor of order $\left(\frac{Z}{137}\right)^2$ where Z is the effective nuclear charge (i.e. the charge interior to the transitioning electron.).

13 Scattering of EM waves by a spherical object

13.1 Plane wave expansion

Suppose an incoming plane wave is scattered by a spherical object. We expect the outgoing waves to be described in terms of vector spherical harmonics. Thus our first task is to expand the incoming wave in terms of the eigenfunctions (8).

We have already found the Green's function, and we may expand it in the limit $r' \gg r$:

$$\frac{e^{ikR}}{4\pi R} = \frac{e^{ik\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|}}{4\pi\left|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\right|} \simeq \frac{e^{ikr'}}{4\pi r'}e^{-i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}}$$

(Note: this is just a mathematical manipulation. We are not yet attaching any physical meaning to $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{x}}'$.) The Green's function expansion (2) may be evaluated with $r_> = r'$, and using the large argument expansion for the Bessel functions in kr' :

$$\begin{aligned}\frac{e^{ikr'}}{4\pi r'}e^{-i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} &= ik\sum_{l,m} j_l(kr)\,h_l^{(1)}(kr')\,Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi') \\ &= ik\sum_{l,m} j_l(kr)\,(-i)^{l+1}\frac{e^{ikr'}}{kr'}Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi')\end{aligned}$$

and the $e^{ikr'}/r'$ term may be brought out of the sum since it does not depend on l or m. Then:

$$e^{-i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} = 4\pi\sum_{l,m} j_l(kr)\,(-i)^l\,Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi')$$

and taking the complex conjugate, we have

$$e^{i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} = 4\pi\sum_{l,m} j_l(kr)\,(i)^l\,Y_{lm}^*(\theta,\phi)\,Y_{lm}(\theta',\phi')$$

Now only the terms in Y_{lm} depend on m. Using the addition theorem (Jackson equation

3.62),

$$\sum_{l,m} Y_{lm}^{*}(\theta,\phi)\, Y_{lm}(\theta',\phi') = \sum_{l} P_l(\cos\gamma)\,\frac{2l+1}{4\pi}$$

where γ is the angle between $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{x}}'$, or equivalently, between $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{k}}$. Finally then:

$$\begin{aligned} e^{i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} &= \sum_l i^l (2l+1)\, j_l(kr)\, P_l(\cos\gamma) \\ &= \sum_l i^l \sqrt{4\pi(2l+1)}\, j_l(kr)\, Y_{l0}(\gamma) \end{aligned}$$

This is exactly the factor that appears in the plane wave expression.

Now we also need to worry about the polarization of the incoming wave. Any wave can be decomposed into either linear or circular polarizations. The circular polarizations turn out to be more convenient here. So write:

$$\tilde{\mathbf{E}}_{\pm} = (\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2)\, e^{ikz} \tag{24}$$

where now we have chosen the $z-$axis along $\tilde{\mathbf{k}}$. Then

$$\bar{\mathbf{B}}_{\pm} = \frac{1}{ik}\tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{E}}_{\pm} = \hat{\boldsymbol{\varepsilon}}_3 \times \tilde{\mathbf{E}}_{\pm} \tag{25}$$

Now we write each field as an expansion in the eigenfunctions:

$$\tilde{\mathbf{E}}_{\pm} = \sum_{l,m} a_{\pm}(l,m)\, j_l \bar{\mathbf{X}}_{lm} + \frac{i}{k} b_{\pm}(l,m)\, \tilde{\boldsymbol{\nabla}}\times\left(j_l \bar{\mathbf{X}}_{lm}\right)$$

and

$$\tilde{\mathbf{B}}_{\pm} = \sum_{l,m} b_{\pm}(l,m)\, j_l \bar{\mathbf{X}}_{lm} - \frac{i}{k} a_{\pm}(l,m)\, \tilde{\boldsymbol{\nabla}}\times\left(j_l \bar{\mathbf{X}}_{lm}\right)$$

where, by the orthogonality of the $\tilde{\mathbf{X}}_{lm}$,

$$a_{\pm}(l,m)\, j_l = \int \tilde{\mathbf{X}}_{lm}^{*} \cdot \tilde{\mathbf{E}}_{\pm} d \tag{26}$$

(We have already proved most of the orthogonality relations we need here in section 8. To show that

$$\bar{\mathbf{X}}_{lm} \cdot \tilde{\boldsymbol{\nabla}}\times\left(j_l \bar{\mathbf{X}}_{lm}\right) = 0$$

expand the curl. The term in $\tilde{\boldsymbol{\nabla}} j_l$ is perpendicular to $\tilde{\mathbf{X}}_{lm}$, and the remaining term involves

$$\bar{\mathbf{X}}_{lm} \cdot \tilde{\boldsymbol{\nabla}} \times \bar{\mathbf{X}}_{lm} = \bar{\mathbf{X}}_{lm} \cdot \left(\tilde{\mathbf{r}}\nabla^2 - \tilde{\boldsymbol{\nabla}}\right) Y_{lm} = -\bar{\mathbf{X}}_{lm} \cdot \tilde{\boldsymbol{\nabla}} Y_{lm}$$

(equation 9) . But $\bar{\mathbf{X}}_{lm} \propto \tilde{\mathbf{r}} \times \tilde{\boldsymbol{\nabla}} Y_{lm}$ is perpendicular to $\tilde{\boldsymbol{\nabla}} Y_{lm}$, and so the relation follows.)

Similarly:

$$b_{\pm}(l,m)\, j_l = \int \tilde{\mathbf{X}}_{lm}^{*} \cdot \tilde{\mathbf{B}}_{\pm} d \tag{27}$$

Now insert the value (24) into (26):

$$\begin{aligned}
a_{\pm}(l,m)\,j_l &= \int \tilde{\mathbf{X}}^*_{lm}\cdot(\hat{\varepsilon}_1 \pm i\hat{\varepsilon}_2)\,e^{ikz}d \\
&= \frac{1}{\sqrt{l(l+1)}}\int L_{\pm}Y^*_{lm}e^{ikz}d \\
&= \frac{\sqrt{l(l+1)-m(m\pm 1)}}{\sqrt{l(l+1)}}\int Y^*_{lm\pm 1}e^{ikz}d \\
&= \frac{\sqrt{l(l+1)-m(m\pm 1)}}{\sqrt{l(l+1)}}\int Y^*_{lm\pm 1}\sum_{l'} i^{l'}\sqrt{4\pi(2l'+1)}j_{l'}(kr)\,Y_{l'0}(\theta)\,d
\end{aligned}$$

Now by orthogonality of the Y_{lm},only $l=l'$ and $m=\mp 1$ contribute:

$$\begin{aligned}
a_{\pm}(l,m)\,j_l &= \frac{\sqrt{l(l+1)-m(0)}}{\sqrt{l(l+1)}}i^l\sqrt{4\pi(2l+1)}j_l(kr)\,\delta_{m,\mp 1} \\
a_{\pm}(l,m) &= i^l\sqrt{4\pi(2l+1)}\delta_{m,\mp 1} \qquad (28)
\end{aligned}$$

with a similar result for the b's. Since $m=\pm 1$, the lowest allowable value of l is 1. Thus we can expand the plane wave fields as:

$$\tilde{\mathbf{E}}_{\pm}(\tilde{\mathbf{x}}) = \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[j_l(kr)\,\tilde{\mathbf{X}}_{l,\pm 1} \pm \frac{1}{k}\tilde{\boldsymbol{\nabla}}\times\left(j_l(kr)\,\tilde{\mathbf{X}}_{l,\pm 1}\right)\right] \qquad (29)$$

where, as we showed in section 10, $m=\pm$ corresponds to ± 1 unit of angular momentum in the $z-$direction. The corresponding result for $\tilde{\mathbf{B}}$ is:

$$\tilde{\mathbf{B}}_{\pm}(\tilde{\mathbf{x}}) = \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[\mp j_l(kr)\,\tilde{\mathbf{X}}_{l,\pm 1} - \frac{i}{k}\tilde{\boldsymbol{\nabla}}\times\left(j_l(kr)\,\tilde{\mathbf{X}}_{l,\pm 1}\right)\right]$$

13.2 Application to scattering

The total electric field is a superposition of the incoming and outgoing fields:

$$\tilde{\mathbf{E}}_{\text{total}} = \tilde{\mathbf{E}}_{\text{incident}} + \tilde{\mathbf{E}}_{\text{scattered}}$$

The scattered waves are ougoing at infinity and so are represented with $h^{(1)}$:

$$\tilde{\mathbf{E}}_{\text{scattered}} = \sum_{l,m} a(l,m)\,h_l^{(1)}\tilde{\mathbf{X}}_{lm} + \frac{i}{k}b(l,m)\,\tilde{\boldsymbol{\nabla}}\times\left(h_l^{(1)}\tilde{\mathbf{X}}_{lm}\right)$$

Now if the scatterer is a conducting sphere, the total tangential electric field on the surface of the sphere must be zero. We have already expanded the curl several times. The result is:

$$\begin{aligned}
\tilde{\boldsymbol{\nabla}}\times\left(h_l^{(1)}\tilde{\mathbf{X}}_{lm}\right) &= \frac{\partial h}{\partial r}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm} + h\tilde{\boldsymbol{\nabla}}\times\tilde{\mathbf{X}}_{lm} \\
&= \frac{\partial h}{\partial r}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm} + \frac{h}{i}\left(\left(\tilde{\mathbf{r}}\nabla^2 - \tilde{\boldsymbol{\nabla}}\right)Y_{lm}\right)
\end{aligned}$$

We only need the tangential part of $\tilde{\boldsymbol{\nabla}}$, which is found from:

$$\hat{\mathbf{r}}\times\left(\tilde{\mathbf{r}}\times\tilde{\boldsymbol{\nabla}}\right)=\tilde{\mathbf{r}}\frac{\partial}{\partial r}-r\tilde{\boldsymbol{\nabla}}$$

and thus

$$\tilde{\boldsymbol{\nabla}}=-i\frac{\hat{\mathbf{r}}}{r}\times\tilde{\mathbf{L}}$$

and so:

$$\tilde{\boldsymbol{\nabla}}\times\left(h_l^{(1)}\tilde{\mathbf{X}}_{lm}\right)=\frac{\partial h}{\partial r}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm}+h\frac{\hat{\mathbf{r}}}{r}\times\tilde{\mathbf{X}}_{lm}+\frac{h}{i}\tilde{\mathbf{r}}\nabla^2Y_{lm}$$

and thus the tangential part of $\tilde{\mathbf{E}}$ is:

$$\tilde{\mathbf{E}}_{\text{scattered}}=\sum_{l,m}a\left(l,m\right)h_l^{(1)}\tilde{\mathbf{X}}_{lm}+\frac{i}{k}b\left(l,m\right)\left(\frac{\partial h_l^{(1)}}{\partial r}+\frac{h_l^{(1)}}{r}\right)\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm}$$

At $r=a$, the surface of the sphere:

$$\begin{aligned}\tilde{\mathbf{E}}_{\text{tan}} &= E_0\sum_{l=1}^{\infty}i^l\sqrt{4\pi\left(2l+1\right)}\left[j_l\left(ka\right)\tilde{\mathbf{X}}_{l,\pm1}\pm\frac{1}{ka}\frac{\partial}{\partial r}\left(rj_l\left(kr\right)\right)\Big|_{r=a}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{l,\pm1}\right]\\ &\quad+\sum_{l,m}a\left(l,m\right)h_l^{(1)}\left(ka\right)\tilde{\mathbf{X}}_{lm}+\frac{i}{ka}b\left(l,m\right)\frac{\partial}{\partial r}\left(rh_l^{(1)}\right)\Big|_{r=a}\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{lm}\\ &= 0\end{aligned}$$

Thus $a\left(l,m\right)$ and $b\left(l,m\right)=0$ unless $m=\pm1$. We can simplify the results by writing:

$$a\left(l,m\right)=i^l\sqrt{4\pi\left(2l+1\right)}\frac{E_0}{2}\alpha_{lm}$$

and

$$b\left(l,m\right)=i^{l-1}\sqrt{4\pi\left(2l+1\right)}\frac{E_0}{2}\beta_{lm}$$

Then $\tilde{\mathbf{E}}_{\text{tan}}=0$ requires:

$$j_l\left(ka\right)+\frac{\alpha_{l,\pm1}}{2}h_l^{(1)}\left(ka\right)=0$$

and thus:

$$\begin{aligned}\alpha_{l,\pm1} &= -2\frac{j_l\left(ka\right)}{h_l^{(1)}\left(ka\right)}=-\frac{h_l^{(1)}\left(ka\right)+h_l^{(2)}\left(ka\right)}{h_l^{(1)}\left(ka\right)}\\ &= -1-\frac{h_l^{(2)}\left(ka\right)}{h_l^{(1)}\left(ka\right)}\end{aligned}$$

and similarly

$$\pm2\frac{\partial}{\partial r}\left(rj_l\left(kr\right)\right)\Big|_{r=a}+\beta_{l,\pm1}\frac{\partial}{\partial r}\left(rh_l^{(1)}\right)\Big|_{r=a}=0$$

So

$$\begin{aligned}\beta_{l,\pm 1} &= \mp \left.\frac{\frac{\partial}{\partial r}\left(rh_l^{(1)}\right)+\frac{\partial}{\partial r}\left(rh_l^{(2)}\right)}{\frac{\partial}{\partial r}\left(rh_l^{(1)}\right)}\right|_{r=a} \\ &= \mp 1 \mp \left.\frac{\frac{\partial}{\partial r}\left(rh_l^{(2)}\right)}{\frac{\partial}{\partial r}\left(rh_l^{(1)}\right)}\right|_{r=a}\end{aligned}$$

We can calculate the radiated power using equation (19). For each polarization separately, we have:

$$\frac{dP_\pm}{d} = \frac{cE_0^2}{8\pi k^2}\left|\sum_l (-i)^{l+1} i^l \sqrt{\pi(2l+1)}\left(\alpha_l \tilde{\mathbf{X}}_{l,\pm 1} + \beta_{l\pm}\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{l,\pm 1}\right)\right|^2$$

and thus the scattering cross section is:

$$\frac{d\sigma_\pm}{d} = \frac{1}{2k^2}\left|\sum_l -i\sqrt{\pi(2l+1)}\left(\alpha_l \tilde{\mathbf{X}}_{l,\pm 1} + \beta_{l\pm}\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{l,\pm 1}\right)\right|^2$$

(Note that the magnitude of the incident field is $\sqrt{2}E_0$).

Now we can compute the fields in the long-wavelength limit. For $ka \ll 1$, $h_l \simeq in_l$, and we have:

$$\begin{aligned}\alpha_l &= -2\frac{j_l(ka)}{h_l^{(1)}(ka)} = -2\frac{(ka)^l}{(2l+1)!!}\frac{(ka)^{l+1}}{(-i(2l-1)!!)} \\ &= \frac{2}{i}\frac{(ka)^{2l+1}}{(2l+1)!!(2l-1)!!}\end{aligned}$$

So clearly $l=1$ is the dominant term. Similarly for $\beta_{l,\pm 1}$ we have:

$$\begin{aligned}\beta_{l,\pm 1} &= \mp 2\frac{\frac{\partial}{\partial r}\left(rj_l(kr)\right)\big|_{r=a}}{i\frac{\partial}{\partial r}\left(rn_l\right)\big|_{r=a}} = \pm\frac{2\frac{\partial}{\partial r}\left(k^l r^{l+1}\right)\big|_{r=a} k^{l+1}}{i(2l+1)!!(2l-1)!!\frac{\partial}{\partial r}\left(r^{-l}\right)\big|_{r=a}} \\ &= \pm\frac{2}{i}\frac{(l+1)(ka)^{2l+1}}{(2l+1)!!(2l-1)!!(-l)} = \mp\frac{l+1}{l}\alpha_l\end{aligned}$$

For $l=1$:

$$\alpha_1 = -\frac{2}{3}i(ka)^3$$

and

$$\beta_{1,\pm 1} = \pm\frac{4}{3}i(ka)^3$$

Then to lowest order:

$$\frac{d\sigma_\pm}{d} = \frac{1}{2k^2}\frac{4}{9}(ka)^6 3\pi\left|\tilde{\mathbf{X}}_{1,\pm 1} \pm 2i\hat{\mathbf{r}} \times \tilde{\mathbf{X}}_{1,\pm 1}\right|^2$$

We have already calculated $\left|\tilde{\mathbf{X}}_{1,\pm 1}\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\right)$. But here we have a cross term:

$$\tilde{\mathbf{X}}_{1,\pm 1}\cdot\left(\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{1,\pm 1}\right)^*$$

From equation (14), we have

$$\tilde{\mathbf{X}}_{1,\pm 1} = \frac{-1}{\sqrt{2}}\left(\sqrt{2\mp 1\left(\pm 1+1\right)}Y_{1,\pm 1+1}\left(\frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{2}\right)+\sqrt{2\mp 1\left(\pm 1-1\right)}Y_{1,\pm 1-1}\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)\pm Y_{1\pm 1}\hat{\mathbf{z}}\right)$$

So:

$$\begin{aligned}
\tilde{\mathbf{X}}_{1,1} &= \frac{-1}{\sqrt{2}}\left(\sqrt{2}Y_{l,0}\left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2}\right)+Y_{1,1}\hat{\mathbf{z}}\right)\\
&= \frac{-1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\sqrt{\frac{3}{4\pi}}\cos\theta\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\hat{\mathbf{z}}\right)\\
&= \frac{-1}{\sqrt{2}}\sqrt{\frac{3}{8\pi}}e^{i\phi}\left(\cos\theta\left(\cos\phi-i\sin\phi\right)\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)-\sin\theta\hat{\mathbf{z}}\right)\\
&= -\frac{1}{4}\sqrt{\frac{3}{\pi}}e^{i\phi}\left(\cos\theta\left(\cos\phi\hat{\mathbf{x}}+i\cos\phi\hat{\mathbf{y}}-i\sin\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}}\right)-\sin\theta\hat{\mathbf{z}}\right)\\
&= -\frac{1}{4}\sqrt{\frac{3}{\pi}}e^{i\phi}\left(i\cos\theta\hat{\boldsymbol{\phi}}-\hat{\boldsymbol{\theta}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{X}}_{1,-1} &= \frac{-1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\sqrt{\frac{3}{4\pi}}\cos\theta\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right)+\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\hat{\mathbf{z}}\right)\\
&= \frac{1}{4}\sqrt{\frac{3}{\pi}}e^{-i\phi}\left(i\cos\theta\hat{\boldsymbol{\phi}}+\hat{\boldsymbol{\theta}}\right)
\end{aligned}$$

So

$$\tilde{\mathbf{X}}_{1,\pm 1} = \frac{1}{4}\sqrt{\frac{3}{\pi}}e^{\pm i\phi}\left(\hat{\boldsymbol{\theta}}\mp i\cos\theta\hat{\boldsymbol{\phi}}\right)$$

Then

$$\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{1,\pm 1} = \frac{1}{4}\sqrt{\frac{3}{\pi}}e^{\pm i\phi}\left(\hat{\boldsymbol{\phi}}\pm i\cos\theta\hat{\boldsymbol{\theta}}\right)$$

And so the cross terms are of the form

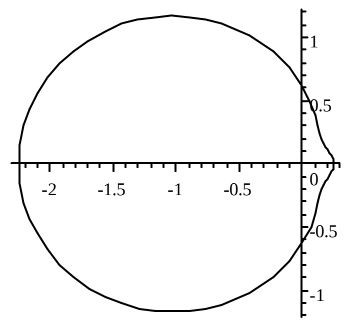
$$\tilde{\mathbf{X}}_{1,\pm 1}\cdot\left(\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{1,\pm 1}\right)^* = \mp i\frac{3}{8\pi}\cos\theta$$

and so

$$\begin{aligned}
\frac{d\sigma_\pm}{d} &= \frac{2}{3}\pi a^2\left(ka\right)^4\left(\tilde{\mathbf{X}}_{1,\pm 1}\pm 2i\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{1,\pm 1}\right)\cdot\left(\tilde{\mathbf{X}}_{1,\pm 1}\pm 2i\hat{\mathbf{r}}\times\tilde{\mathbf{X}}_{1,\pm 1}\right)^*\\
&= \frac{2}{3}\pi a^2\left(ka\right)^4\left(5\frac{3}{16\pi}\left(1+\cos^2\theta\right)-2\frac{3}{4\pi}\cos\theta\right)\\
&= a^2\left(ka\right)^4\left(\frac{5}{8}\left(1+\cos^2\theta\right)-\cos\theta\right)
\end{aligned}$$

which agrees with equation 9.94.

This result shows explicitly the interference between the modes.



scattering cross section versus angle