Additional notes on Cerenkov.

Calculating the fields when λ is imaginary.

Strictly, we need to recalculate the fields because the results we used are valid only for $\operatorname{Re}\lambda > 0$. The result is unchanged through equation (18). Then suppose $\lambda = i\mu$ where $\mu = \frac{\omega}{v}\sqrt{\beta^2\varepsilon - 1}$ and μ is real.

First note what happens to the integral over k_z. The result is unchanged if $|k_y| > \mu$. But if $|k_y| < \mu$ we have

$$\int_{-\infty}^{+\infty} \frac{dk_z}{k_y^2 + k_z^2 - \mu^2} = 2\int_0^{+\infty} \frac{dk_z}{k_z^2 - \left(\mu^2 - k_y^2\right)}$$

we integrate by letting $k_z = \sqrt{\mu^2 - k_y^2}\cosh\theta$.

$$\begin{aligned}
I &= 2\int \frac{\sqrt{\mu^2 - k_y^2}\sinh\theta d\theta}{\left(\mu^2 - k_y^2\right)\left(\sinh^2\theta\right)} = 2\frac{1}{\sqrt{\mu^2 - k_y^2}}\int \frac{\sinh\theta}{1 - \cosh^2\theta} d\theta \\
&= \frac{-2}{\sqrt{\mu^2 - k_y^2}}\frac{1}{2}\int \left(\frac{\sinh\theta}{\cosh\theta - 1} - \frac{\sinh\theta}{\cosh\theta + 1}\right) d\theta \\
&= -\frac{1}{\sqrt{\mu^2 - k_y^2}}\ln\left(\frac{\cosh\theta - 1}{\cosh\theta + 1}\right)
\end{aligned}$$

Putting in the limits for k_z, we have

$$\begin{aligned}
I &= \frac{-1}{\sqrt{\mu^2 - k_y^2}}\ln\frac{k_z - \sqrt{\mu^2 - k_y^2}}{k_z + \sqrt{\mu^2 - k_y^2}}\Bigg|_0^\infty \\
&= -\frac{1}{\sqrt{\mu^2 - k_y^2}}\left[\ln 1 - \ln(-1)\right] = \frac{0 \pm i\pi}{\sqrt{\mu^2 - k_y^2}} \qquad k_y < \mu
\end{aligned}$$

Note that the sign here depends on the branch we take for the log function.

Then for E_y we need (notes 3 page 11)

$$E_y = \frac{-i}{(2\pi)^{3/2}}\frac{2}{\varepsilon}\frac{Ze}{v}\left\{\left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty}\right) dk_y \frac{\pi k_y}{\sqrt{\lambda^2 + k_y^2}}\exp\left(ibk_y\right) + \int_{-\mu}^{+\mu} dk_y \frac{\pm i\pi k_y}{\sqrt{\mu^2 - k_y^2}}\exp\left(ibk_y\right)\right\}$$

The term in curly brackets is

$$\frac{\{\}}{\pi} = 2i\int_{\mu}^{+\infty} dk_y \frac{k_y \sin\left(bk_y\right)}{\sqrt{k_y^2 - \mu^2}} \pm i\left(2i\right)\int_0^{\mu} dk_y \frac{k_y \sin\left(bk_y\right)}{\sqrt{\mu^2 - k_y^2}}$$

These two pieces are in GR: 3.771 # 10 and # 11 with $a = b$ and $v = 0$. Thus

$$\frac{\{\}}{\pi} = 2\frac{\sqrt{\pi}}{2}\mu\left[\mp\Gamma\left(\frac{1}{2}\right)J_1\left(b\mu\right) + i\lim_{\nu\to 0}\Gamma\left(\nu+\frac{1}{2}\right)\left(\frac{2\mu}{b}\right)^{\nu}N_{-\nu-1}\left(b\mu\right)\right]$$

Using the result that $N_{-m}\left(x\right) = \left(-1\right)^m N_m\left(x\right)$ and $\Gamma\left(1/2\right) = \sqrt{\pi}$, we have

$$\begin{aligned}
\{\} &= \pi^2\mu\left[\mp J_1\left(b\mu\right) + i\lim_{\nu\to 0}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}}\left(\frac{2\mu}{b}\right)^{\nu}N_{-\nu-1}\left(b\mu\right)\right] \\
&= \mp\pi^2\mu\left[J_1\left(b\mu\right) \pm iN_1\left(b\mu\right)\right] \\
&= \mp\pi^2\mu H_1^{(1,2)}\left(b\mu\right)
\end{aligned}$$

For large argument, the Hankel functions also have exponential form, so we have (choosing H^1 for outgoing waves)

$$\begin{aligned}
E_y &= \frac{i}{\left(2\pi\right)^{3/2}}\frac{2}{\varepsilon}\frac{Ze}{v}\pi^2\mu H_1^{(1)}\left(b\mu\right) \\
&\to i\sqrt{\frac{\pi}{2}}\frac{Ze}{\varepsilon v}\mu\sqrt{\frac{2}{\pi b\mu}}\exp\left(i\left(b\mu - \frac{\pi}{2} - \frac{\pi}{4}\right)\right) \quad \text{for large } b\mu \\
&= \frac{Ze}{\varepsilon v}\sqrt{\frac{\mu}{b}}\exp\left(i\left(b\mu - \frac{\pi}{4}\right)\right) \\
&= \frac{Ze}{\varepsilon v}\sqrt{\frac{\omega}{vb}\sqrt{\beta^2\varepsilon - 1}}\exp\left(i\left(b\mu - \frac{\pi}{4}\right)\right) \qquad (1)
\end{aligned}$$

Similarly for E_x we have (Notes 3 pg 12)

$$\frac{-i\omega}{\left(2\pi\right)^{1/2}}\left(1-\beta^2\varepsilon\right)\frac{Ze}{\varepsilon v^2}I$$

where this integral is

$$\begin{aligned}
I &= \left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty}\right)dk_y\frac{1}{\sqrt{\lambda^2 + k_y^2}}\exp\left(ibk_y\right) + \int_{-\mu}^{+\mu}dk_y\frac{\mp i}{\sqrt{\mu^2 - k_y^2}}\exp\left(ibk_y\right) \\
&= 2\int_{\mu}^{+\infty}dk_y\frac{\cos bk_y}{\sqrt{k_y^2 - \mu^2}} \mp 2i\int_0^{\mu}dk_y\frac{\cos bk_y}{\sqrt{\mu^2 - k_y^2}}
\end{aligned}$$

GR 3.771#8 and 9 to the rescue!

$$\begin{aligned}
I &= 2\frac{\sqrt{\pi}}{2}\Gamma\left(1/2\right)\left[\mp iJ_0\left(b\mu\right) - N_0\left(b\mu\right)\right] \\
&= \mp i\pi\left[J_0\left(b\mu\right) \mp iN_0\left(b\mu\right)\right] \\
&= \pm i\pi H_0^{(1,2)}\left(b\mu\right)
\end{aligned}$$

and then, choosing $H^{(1)}$ again,

$$\begin{aligned}
E_x &= \frac{i\omega}{(2\pi)^{1/2}}\left(\beta^2\varepsilon-1\right)\frac{Ze}{\varepsilon v^2}i\pi H_0^{(1)}(b\mu) \\
&\to \frac{-\omega}{(2\pi)^{1/2}}\left(\beta^2\varepsilon-1\right)\frac{Ze}{\varepsilon v^2}\pi\sqrt{\frac{2}{\pi b\mu}}\exp\left[i\left(b\mu-\frac{\pi}{4}\right)\right] \quad \text{for large } b\mu \\
&= -\omega\left(\beta^2\varepsilon-1\right)\frac{Ze}{\varepsilon v^2}\sqrt{\frac{v}{b\omega\sqrt{\beta^2\varepsilon-1}}}\exp\left(i\left(b\mu-\frac{\pi}{4}\right)\right) \qquad (2)
\end{aligned}$$

-and finally

$$\begin{aligned}
B_z &= \beta\varepsilon E_y \\
&= \beta\frac{Ze}{v}\sqrt{\frac{\mu}{b}}\exp\left(i\left(b\mu-\frac{\pi}{4}\right)\right) \qquad (3)
\end{aligned}$$

Transforming back, we can see the wave form:

$$\begin{aligned}
E_x(b,t) &= -\frac{Ze}{v^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\omega}{\varepsilon}\left(\beta^2\varepsilon-1\right)\sqrt{\frac{1}{b\mu}}\exp\left(i\left(b\mu-\frac{\pi}{4}\right)\right)e^{-i\omega t}d\omega \\
&= -\frac{Ze}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\omega}{\varepsilon v}\left(\beta^2\varepsilon-1\right)\sqrt{\frac{1}{b\mu}}\exp\left(i\left(b\mu-\frac{\pi}{4}\right)\right)e^{-i\omega t}\frac{d\omega}{v}
\end{aligned}$$

Remember that $\mu=\frac{\omega}{v}\sqrt{\beta^2\varepsilon-1}$ and ε also depends on ω. This expression shows the wave form through the exponential, and is dimensionally correct.

The Poynting flux is

$$\begin{aligned}
\vec{S} &= \frac{c}{4\pi}\left(-E_xB_z\hat{y}+E_yB_z\hat{x}\right) \\
&= \frac{c\beta\varepsilon}{4\pi}\left(-E_xE_y\hat{y}+E_y^2\hat{x}\right)
\end{aligned}$$

When we integrate to get the energy transmitted to large b we find (eqns2 and 1)

$$\begin{aligned}
\frac{d\mathcal{E}}{dx} &= -\frac{c\beta}{4\pi}\int_{-\infty}^{\infty}\varepsilon E_x(\omega)E_y^*(\omega)\,d\omega 2\pi b \\
&= \frac{c\beta}{2}b\int_{-\infty}^{\infty}\left[\omega\left(\beta^2\varepsilon-1\right)\varepsilon\frac{Ze}{\varepsilon v^2}\sqrt{\frac{1}{b\mu}}\exp\left(i\left(b\mu-\frac{\pi}{4}\right)\right)\right]\left[\pi\frac{Ze}{\varepsilon v}\sqrt{\frac{\mu}{b}}\exp\left(i\left(b\mu-\frac{\pi}{4}\right)\right)\right]^*d\omega \\
&= \frac{v}{2}\frac{(Ze)^2}{v^3}\int_{-\infty}^{\infty}\frac{\omega}{\varepsilon}\left(\beta^2\varepsilon-1\right)d\omega \\
&= \frac{(Ze)^2}{2c^2}\int_{-\infty}^{\infty}\omega\left(1-\frac{1}{\beta^2\varepsilon}\right)d\omega
\end{aligned}$$

Note the result is real and positive, and agrees with eqn(22) in notes 3.