Example using spherical harmonics– Sp 2020

Magnetic field due to a current loop.

A circular loop of radius a carries current I. We place the origin at the center of the loop, with polar axis perpendicular to the plane of the loop. Then the current density is

$$\vec{j} = I\frac{\delta(r-a)}{a}\delta(\mu)\ \hat{\phi}$$

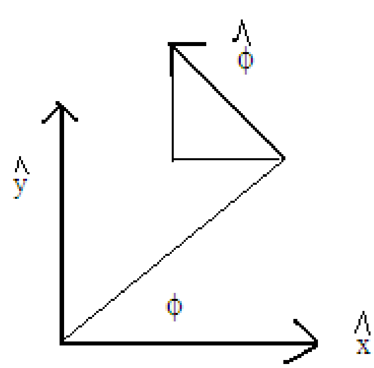
(You can get this most easily by starting with the expression in cylindrical coordinates

$$\vec{j} = I\delta(z)\,\delta(r-a)\,\hat{\phi}$$

and using $z = r\cos\theta$. See also Lea pg 315 Example 6.7.) Then the magnetic vector potential is (Notes 1 eqn 21)

$$\begin{aligned}\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}' \qquad (1)\\ &= \frac{\mu_0}{4\pi a}I\int\frac{\delta(r'-a)\,\delta(\mu')\ \hat{\phi}(\phi')}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\end{aligned}$$

We must take care here, because the unit vector $\hat{\phi}$ is not a constant. We must re-express it in terms of the constant Cartesian unit vectors,



$$\hat{\phi}(\phi') = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$$

and thus:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi a}I\int\frac{\delta(r'-a)\,\delta(\mu')}{|\vec{x}-\vec{x}'|}\left(-\sin\phi'\hat{x} + \cos\phi'\hat{y}\right)\ d^3\vec{x}'$$

Now we use our **most useful result** (J eqn 3.70, spherprobnotes eqn 25) to expand the $1/|\vec{x}-\vec{x}'|$ in the integrand. With $r_< = \min(r, r')$ and similarly for $r_>$,

$$\begin{aligned}
\vec{A}(\vec{x}) &= \frac{\mu_0 I}{4\pi a}\int \delta(r'-a)\,\delta(\mu') \\
&\qquad \times \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi')\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)d^3\vec{x}' \\
&= \frac{\mu_0 I}{4\pi a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\times \\
&\qquad \int_0^{2\pi}\int_{-1}^{+1}\int_0^{\infty}\delta(r'-a)\,\delta(\mu')\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)(r')^2\,dr'd\mu'd\phi' \\
&= \mu_0 a I\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\phi)}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int_0^{2\pi}Y_{lm}^*\left(\frac{\pi}{2},\phi'\right)\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)\,d\phi'
\end{aligned}$$

where we used the sifting property to evaluate the integrals over r' and μ'. We now interpret $r_<$ as the lesser of r and a, and similarly for $r_>$.

To do the integral over ϕ', we rewrite the sines and cosines in terms of exponentials:

$$\begin{aligned}
I_{lm} &= \int_0^{2\pi}Y_{lm}^*\left(\frac{\pi}{2},\phi'\right)\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)d\phi' \\
&= \int_0^{2\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\,e^{-im\phi'}\left[e^{i\phi'}\left(\frac{\hat{y}+i\hat{x}}{2}\right)+e^{-i\phi'}\left(\frac{\hat{y}-i\hat{x}}{2}\right)\right]d\phi'
\end{aligned}$$

The integral is zero unless $m=\pm 1$. (Be alert here– if you use the mantra "axisymmetry so $m=0$" you will get into big trouble!) With $m=+1$ we get:

$$I_{l1} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}}P_l^1(0)\left(\frac{\hat{y}+i\hat{x}}{2}\right)2\pi$$

and with $m=-1$ and $Y_{l,-1} = -Y_{l1}^*$ (spherprob notes eqn 24)

$$\begin{aligned}
I_{l,-1} &= -I_{l1}^* \\
&= -\pi\sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}}P_l^1(0)\left(\hat{y}-i\hat{x}\right)
\end{aligned}$$

So

$$\vec{A}(\vec{x}) = \mu_0 a I\sum_{l=1}^{\infty}\frac{\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}}P_l^1(0)\left[(\hat{y}+i\hat{x})\,Y_{l1}(\theta,\phi)-(\hat{y}-i\hat{x})\,Y_{l,-1}\right]$$

The sum over l starts at 1 now because with $l = 0$ there is no $m = \pm 1$ term. Then

$$\begin{aligned}
\vec{A}(\vec{x}) &= \mu_0 a I \sum_{l=1}^{\infty} \frac{\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \frac{2l+1}{4\pi} \frac{(l-1)!}{(l+1)!} P_l^1(0) P_l^1(\mu) \left[(\hat{y} + i\hat{x}) e^{i\phi} + (\hat{y} - i\hat{x}) e^{-i\phi}\right] \\
&= \frac{\mu_0 a I}{4} \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} \frac{1}{l(l+1)} P_l^1(0) P_l^1(\mu) (2\hat{y}\cos\phi - 2\hat{x}\sin\phi) \\
&= \frac{\mu_0 a I}{2} \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} \frac{P_l^1(0) P_l^1(\mu)}{l(l+1)} \hat{\phi} \qquad (2)
\end{aligned}$$

We might have expected to find that $\vec{A}$ is in the ϕ direction, parallel to $\vec{j}$. Check dimensions: $\vec{A}$ is current times μ_0, which is consistent with (1).

We can simplify a bit by inserting the value of $P_l^1(0)$. First note that P_l is even if l is even, and odd if l is odd. Since P_l^m is proportional to the mth derivative of P_l, P_l^m will be odd if $l+m$ is odd and even if $l+m$ is even. (Lea pg 385) So $P_l^m(0) = 0$ unless $l+m$ is even, or, in this case, l is odd. Note that this result is due to the reflection anti-symmetry of $\vec{B}$ about the plane of the ring.

Now we can use the recursion relation (J3.29 or Lea 8.37)

$$\begin{aligned}
l P_l(\mu) &= \mu P_l'(\mu) - P_{l-1}'(\mu) \\
l P_l(0) &= -P_{l-1}'(0) = P_{l-1}^1(0) \qquad \text{(Lea 8.53)}
\end{aligned}$$

Thus, using Lea 8.47, with $l = 2n+1$,

$$\begin{aligned}
P_l^1(0) &= (l+1) P_{l+1}(0) = (l+1)(-1)^{(l+1)/2} \frac{l!!}{(l+1)!!} \\
P_{2n+1}^1(0) &= (-1)^{n+1} \frac{(2n+1)!!}{(2n)!!} \qquad (n > 0) \\
&= (-1)^{n+1} \frac{(2n+1)!!}{2^n n!}
\end{aligned}$$

Note that $P_1^1(\mu) = -\sqrt{1-\mu^2} = -\sin\theta$ (Lea Table 8.1), so $P_1^1(0) = -1$. Thus (2) becomes

$$\begin{aligned}
\vec{A}(\vec{x}) &= \frac{\mu_0 a I}{2} \left[\frac{\sin\theta}{2r_>} + \sum_{n=1}^{\infty} \frac{r_<^{2n+1}}{r_>^{2n+2}} \frac{1}{(2n+1)(2n+2)} (-1)^{n+1} \frac{(2n+1)!!}{2^n n!} P_{2n+1}^1(\mu)\right] \hat{\phi} \\
&= \frac{\mu_0 a I}{4 r_>} \left[\frac{\sin\theta}{r_>} - \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^{2n+1} \frac{(2n+1)!!}{(n+1)(2n+1)} \frac{(-1)^n}{2^n n!} P_{2n+1}^1(\mu)\right] \hat{\phi} \qquad (3)
\end{aligned}$$

Outside the loop, $r > a$, and

$$\vec{A}(\vec{x}) = \frac{\mu_0 a I}{4r} \left[\frac{a}{r} \sin\theta - \sum_{n=1}^{\infty} (-1)^n \left(\frac{a}{r}\right)^{2n+1} \frac{(2n-1)!!}{2^n (n+1)!} P_{2n+1}^1(\mu)\right] \hat{\phi} \qquad (4)$$

For $r \gg a$, $n = 0$ ($\ell = 1$) is the dominant term:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0 I}{4} \frac{a^2}{r^2} \sin\theta \; \hat{\phi}$$

so

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0 a^2 I}{4r^2} \sin\theta \; \hat{\phi} = \frac{\mu_0}{4\pi} \frac{m}{r^2} \sin\theta \; \hat{\phi} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \tag{5}$$

where $\vec{m} = \pi a^2 I \; \hat{z}$ is the magnetic moment of the loop. Compare with Jackson equation 5.55. Then, in this limit

$$\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} = \frac{\hat{r}}{r \sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\mu_0}{4\pi} \frac{m}{r^2} \sin^2\theta \right) - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} \frac{\mu_0}{4\pi} \frac{m}{r} \sin\theta \\
&= \frac{\hat{r}}{r \sin\theta} \frac{\mu_0}{4\pi} \frac{m}{r^2} 2 \sin\theta \cos\theta + \frac{\hat{\theta}}{r} \frac{\mu_0}{4\pi} \frac{m}{r^2} \sin\theta \\
&= \frac{\mu_0}{4\pi} \frac{m}{r^3} \left(\hat{r} \; 2\cos\theta + \hat{\theta} \; \sin\theta \right)
\end{aligned}$$

This is a dipole field, as expected.

Inside the loop, $r < a$ and we have:

$$\vec{A}(\vec{x}) = \frac{\mu_0 a I}{4a} \left[\frac{r}{a} \sin\theta - \sum_{n=1}^{\infty} (-1)^n \left(\frac{r}{a} \right)^{2n+1} \frac{(2n-1)!!}{2^n (n+1)!} P_{2n+1}^1(\mu) \right] \hat{\phi} \tag{6}$$

Near the center, $r \ll a$, the $n = 0$ term dominates again, and we have:

$$\vec{A} \simeq \frac{\mu_0 I}{4} \frac{r}{a} \sin\theta \; \hat{\phi}$$

and

$$\begin{aligned}
\vec{B}(\vec{x}) &\simeq \frac{\mu_0}{4a} I \left[\frac{\hat{r}}{r \sin\theta} \frac{\partial}{\partial\theta} \left(r \sin^2\theta \right) - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} r^2 \sin\theta \right] \\
&= \frac{\mu_0}{2a} I \left(\hat{r} \cos\theta - \hat{\theta} \sin\theta \right) \\
&= \frac{\mu_0}{2a} I \; \hat{z}
\end{aligned} \tag{7}$$

a uniform field, as expected. Compare with Lea and Burke equation 28.7 with $z = 0$.

Field on axis:
From LB 28.7, the field on the polar ($z-$) axis is

$$\vec{B}(z) = \frac{\mu_0 I a^2}{2\left(z^2 + a^2\right)^{3/2}}\hat{z}$$

So for $z > a$ we can do a binomial expansion to get

$$\begin{aligned}
\vec{B}(z) &= \frac{\mu_0 I a^2}{2z^3}\left(1 - \frac{3}{2}\frac{a^2}{z^2} + \left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\frac{1}{2}\frac{a^4}{z^4} + \cdots\right)\hat{z} \\
&= \frac{\mu_0 I a^2}{2z^3}\left(1 - \frac{3}{2}\frac{a^2}{z^2} + \frac{15}{8}\frac{a^4}{z^4} + \cdots\right)\hat{z} \qquad (8)
\end{aligned}$$

We can also find $\vec{B}$ from the solution (4) for $r > a$:

$$\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
&= \frac{\mu_0 a I}{4}\left\{\frac{\hat{r}}{r \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[\frac{a}{r^2}\sin\theta + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{a^{2n+1}}{r^{2n+2}}\frac{(2n-1)!!}{2^n(n+1)!}P_{2n+1}^1(\mu)\right]\right)\right. \\
&\qquad \left. -\frac{\hat{\theta}}{r}\left[\frac{\partial}{\partial r}\left(\frac{a}{r}\sin\theta + \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{a}{r}\right)^{2n+1}\frac{(2n-1)!!}{2^n(n+1)!}P_{2n+1}^1(\mu)\right)\right]\right\}
\end{aligned}$$

For $\mu = 1$ $(\theta = 0)$, the theta component is $\frac{\mu_0 a I}{4r}$ times

$$\left(\frac{a}{r^2}\sin\theta + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n+1)}{r}\left(\frac{a}{r}\right)^{2n+1}\frac{(2n-1)!!}{2^n(n+1)!}P_{2n+1}^1(1)\right)$$

But

$$P_{2n+1}^1(1) = -\sqrt{1-\mu^2}\frac{d}{d\mu}P_{2n+1}(\mu)\bigg|_{\mu=1} = 0$$

So the theta component on axis is zero, as we would expect from the symmetry (azimuthal symmetry about the axis and reflection anti-symmetry about the plane of the loop).

The $r-$component is

$$\begin{aligned}
B_r &= \frac{\mu_0 a I}{4}\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[\frac{a}{r^2}\sin\theta - \sum_{n=1}^{\infty}(-1)^n\frac{a^{2n+1}}{r^{2n+2}}\frac{(2n-1)!!}{2^n(n+1)!}P_{2n+1}^1(\mu)\right]\right) \\
&= \frac{\mu_0 a^2 I}{4r^3}\hat{r}\left\{2\cos\theta + \frac{\partial}{\partial\mu}\left[\sqrt{1-\mu^2}\sum_{n=1}^{\infty}(-1)^n\frac{a^{2n}}{r^{2n}}\frac{(2n-1)!!}{2^n(n+1)!}P_{2n+1}^1(\mu)\right]\right\}
\end{aligned}$$

From the definition of P_l^1 (Lea 8.53) and the differential equation for P_{2n+1} (Lea 8.19), we find

$$\begin{aligned}
\frac{d}{d\mu}\left[\sqrt{1-\mu^2}P_{2n+1}^1(\mu)\right] &= -\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d}{d\mu}P_{2n+1}(\mu)\right] \\
&= (2n+1)(2n+2)P_{2n+1}(\mu)
\end{aligned}$$

Then we evaluate at $\mu = 1$ where $P_{2n+1}(1) = 1$

$$B_r(r,0) = \frac{\mu_0 a^2 I}{4}\frac{\hat{r}}{r^3}\left[2 + \sum_{n=1}^{\infty}(-1)^n \frac{a^{2n}}{r^{2n}}\frac{(2n-1)!!}{2^n(n+1)!}(2n+1)(2n+2)\right]$$

Changing to the $z-$coordinate, we get

$$\begin{aligned} B_z(z) &= \frac{\mu_0 a^2 I}{2z^3}\hat{z}\left(1 + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \frac{a^{2n}}{z^{2n}}\frac{(2n+1)!!}{2^{n-1}n!}\right) \\ &= \frac{\mu_0 a^2 I}{2z^3}\hat{z}\left(1 - \frac{3}{2}\frac{a^2}{z^2} + \frac{a^4}{2z^4}\frac{5\times 3}{2\times 2} + \cdots\right) \\ &= \frac{\mu_0 a^2 I}{2z^3}\hat{z}\left(1 - \frac{3}{2}\left(\frac{a}{z}\right)^2 + \frac{15}{8}\left(\frac{a}{z}\right)^4 + \cdots\right) \end{aligned}$$

which agrees with (8).