Motion using Hamiltonians

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Let's look at a particle moving in a uniform magnetic field $\vec{B} = B_0\hat{z}$. We can satisfy the Lorentz gauge condition
$$\partial_\alpha A^\alpha = 0$$
and the relation $\vec{B} = \vec{\nabla} \times \vec{A}$ with the potential
$$A^\alpha = (0, 0, B_0 x, 0)$$
The corresponding canonical momentum is:
$$P^\alpha = p^\alpha + \frac{q}{c}A^\alpha = \left(p^0, p^1, p^2 + \frac{q}{c}B_0 x, p^3\right)$$
and Hamilton's equations are:
$$\frac{dP^\alpha}{d\tau} = \frac{q}{mc}\left(P_\beta - \frac{q}{c}A_\beta\right)\partial^\alpha A^\beta \tag{1}$$
where the only non-zero component of $\partial^\alpha A^\beta$ is $\partial^1 A^2 = -B_0$, since $\partial^1 \equiv -\frac{\partial}{\partial x}$. Then from Hamilton's equations (1), we get:
$$\frac{dP^0}{d\tau} = 0 \Rightarrow P^0 = p^0 = \gamma mc = \text{ constant} \tag{2}$$
Thus the particle's energy remains constant.
$$\frac{dP^2}{d\tau} = 0 \Rightarrow P^2 = p^2 + \frac{q}{c}B_0 x = mu_y + \frac{q}{c}B_0 x = \text{ constant}$$
where u_y is the $y-$component of the 4-velocity, = γv_y and $\vec{v}$ is the 3-velocity. We may choose our origin so that $u_y = 0$ when $x = 0$, and then:
$$u_y = -\frac{qB_0}{mc}x \tag{3}$$
The next equation is:
$$\frac{dP^3}{d\tau} = 0 \Rightarrow P^3 = p^3 = mu_z = \gamma m v_z = \text{ constant}$$
Thus the particle's velocity component along the field remains constant. The last equation is:
$$\begin{aligned} \frac{dP^1}{d\tau} &= \frac{q}{c}(u_2)(-B_0) \\ \frac{dp^1}{d\tau} &= \frac{d}{d\tau}\left(mu^1\right) = \frac{q}{c}(-u_y)(-B_0) = \frac{q}{c}u_y B_0 \end{aligned}$$

where $u_y = u^2 = -u_2$. We may insert the result (3) on the right hand side:

$$\frac{du^1}{d\tau} = \frac{d^2x}{d\tau^2} = -\left(\frac{qB_0}{mc}\right)^2 x$$

which we may integrate immediately to get:

$$x = A\cos\Omega\tau + B\sin\Omega\tau$$

with Ω equal to the cyclotron frequency:

$$\Omega = \frac{qB_0}{mc}$$

Then from equation (3), we have

$$\frac{dy}{d\tau} = u_y = -\Omega\left(A\cos\Omega\tau + B\sin\Omega\tau\right)$$

and thus

$$y = -A\sin\Omega\tau + B\cos\Omega\tau + C$$

Since we have already established that γ remains constant (eqn 2), we may write $\tau = t/\gamma$ We may also choose the origin of τ (and t) so that:

$$x = A\cos\frac{\Omega t}{\gamma}$$

(i.e. $B = 0$), and again, careful choice of origin allows us to take $C = 0$,so

$$y = -A\sin\frac{\Omega t}{\gamma}$$

which is circular motion with angular frequency $\Omega/\gamma = eB/\gamma mc$ and radius A. No surprises.

It is sometimes convenient to write the angular velocity as a vector. Remember that $\vec{\omega}$ points along the axis of rotation, per the RHR. This particle is gyrating with y decreasing, so $\vec{\Omega}$ is in the negative $z-$direction, and thus:

$$\vec{\Omega} = -\frac{q}{mc}\vec{B}$$