Grad B Drift

In a single frame with only a magnetic field $\vec{B} = B(x)\,\hat{z}$ we have the equation of motion:

$$\frac{d\vec{p}}{dt} = q\frac{\vec{v}}{c} \times \vec{B}$$

or, since γ is constant (Hamilton notes eqn 2),

$$\frac{d\vec{v}}{dt} = -\frac{q}{\gamma mc}\vec{B} \times \vec{v}$$

We expect the result to be a gyration plus constant drift, so the equation of motion should take the form

$$\frac{d}{dt}(\vec{v} - \vec{v}_D) = \vec{\omega} \times (\vec{v} - \vec{v}_D) \tag{1}$$

With the usual result

$$\vec{\omega} = -\frac{q}{\gamma mc}\vec{B}_0 \tag{2}$$

we may rewrite our equation by Taylor expanding $\vec{B}$:

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{d}{dt}(\vec{v} - \vec{v}_D) = -\frac{q}{\gamma mc}\left(\vec{B}_0 + \left[(\vec{x} - \vec{x}_0)\cdot\vec{\nabla}\right]\vec{B}\Big|_0 + \cdots\right) \times \vec{v} \\ &= \vec{\omega} \times \vec{v} - \frac{\omega}{B_0}\left[(\vec{x} - \vec{x}_0)\cdot\vec{\nabla}\right]\vec{B}\Big|_0 \times \vec{v} + \cdots \end{aligned} \tag{3}$$

Now we make the assumption that the field is slowly varying in the sense that

$$\frac{B}{\left|\vec{\nabla}B\right|} \gg r_l = \frac{|(\vec{v} - \vec{v}_D)|}{\omega}$$

so we may drop higher terms in the expansion. We time average to obtain the drift., since

$$\vec{v} = \vec{\omega} \times \vec{r} + \vec{v}_D$$

and the gyrational part has components that oscillate in time. Similarly

$$\vec{x} = \vec{x}_0 + \vec{r} + \vec{v}_D t$$

where $\vec{r}$ has components that oscilate in time. Thus the second term on the right hand side of (3) is:

$$\begin{aligned}-\vec{\omega} \times \vec{v}_D &= -< \frac{\omega}{B_0}\left[(\vec{x} - \vec{x}_0)\cdot\vec{\nabla}\right]\vec{B}\Big|_0 \times \vec{v} > \\ &= < \vec{\omega} \times \left(\frac{1}{B_0}\left[(\vec{x} - \vec{x}_0)\cdot\vec{\nabla}\right] B|_0\right)\vec{v} >\end{aligned}$$

Now we time average. Only the oscillating part of $\vec{v}$ contributes, so we get:

$$\vec{v}_D = -< \frac{1}{B_0}\left(\left[(\vec{x} - \vec{x}_0)\cdot\vec{\nabla}\right] B|_0\right)\vec{\omega} \times (\vec{x} - \vec{x}_0) >$$

The operator

$$\left[(\vec{x}-\vec{x}_0)\cdot\vec{\nabla}\right]\,\vec{B}\Big|_0$$

selects out only the component of $\vec{x}-\vec{x}_0$ parallel to the gradient of $\vec{B}$, the x component here, and so only one component of $\vec{\omega}\times\vec{r}$ in $\vec{v}$ will contribute to the time average.

$$\begin{aligned}
\vec{v}_D &= -<\frac{1}{B_0}\,(x-x_0)^2\,\vec{\omega}\times\vec{\nabla}B> \quad = -\frac{r_l^2}{2B_0}\vec{\omega}\times\vec{\nabla}B \\
&= \frac{\omega r_l^2}{2B_0^2}\vec{B}_0\times\vec{\nabla}B
\end{aligned}$$

which is Jackson's result.