# 1 Lagrangian for a continuous system

Let's start with an example from mechanics to get the big idea. The physical system of interest is a string of length $L$ and mass per unit length $\mu$ fixed at both ends, and under tension $T$. Choose $x-$axis along the unperturbed string, and $y-$axis perpendicular to it. When the string is vibrating, its kinetic energy is:

$$T = \int_0^L \frac{1}{2} v^2 dm = \int_0^L \frac{1}{2} \left( \frac{\partial y}{\partial t} \right)^2 \mu dx = \int_0^L \frac{1}{2} \left( \dot{y} \right)^2 \mu dx$$

To get the potential energy, we use the method of virtual work. The net force on a string segment has components:

$$dF_x = T \cos\theta_1 - T \cos\theta_2 \simeq 0$$

and

$$\begin{aligned} dF_y &= T \sin\theta_1 - T \sin\theta_2 \approx T \tan\theta_1 - T \tan\theta_2 = T \left( \left. \frac{\partial y}{\partial x} \right|_{x+dx} - \left. \frac{\partial y}{\partial x} \right|_x \right) \\ &= T \frac{\partial^2 y}{\partial x^2} dx \end{aligned}$$

Then the virtual work is

$$\delta W = \int_0^L dF_y \delta y = \int_0^L T \frac{\partial^2 y}{\partial x^2} dx \delta y$$

Now integrate by parts, and make use of the fixed end condition:

$$\delta W = T \left[ \delta y y' \big|_0^L - \int_0^L \delta \left( y' \right) y' dx \right] = -\frac{T}{2} \int_0^L \delta \left[ \left( y' \right)^2 \right] dx$$

Then if $\vec{F} = -\vec{\nabla} V$, then $\vec{F} \cdot d\vec{s} = -\delta V$ and $V = -\int \vec{F} \cdot ds$. Here

$$V = -\int \delta W = \frac{T}{2} \int_0^L \left( y' \right)^2 dx$$

The Lagrangian for the string is:

$$L = T - V = \int_0^L \frac{1}{2} \left[ \mu \left( \dot{y} \right)^2 - T \left( y' \right)^2 \right] dx$$

where

$$\mathcal{L} = \frac{1}{2} \left[ \mu \left( \dot{y} \right)^2 - T \left( y' \right)^2 \right] \tag{1}$$

is the Lagrangian density for the string.

The action is

$$A = \int L dt = \int_{t_1}^{t_2} \int_0^L \frac{1}{2} \left[ \mu \left(\dot{y}\right)^2 - T \left(y'\right)^2 \right] dx dt$$

Taking the variation of the action, we get

$$\delta A = \int_{t_1}^{t_2} \int_0^L \left[ \frac{\partial \mathcal{L}}{\partial \dot{y}} \delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'} \delta y' + \frac{\partial \mathcal{L}}{\partial y} \delta y \right] dx dt$$

Integrating by parts gives:

$$\delta A = \int_{t_1}^{t_2} \int_0^L \left[ -\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial y'} + \frac{\partial \mathcal{L}}{\partial y} \right] \delta y dx dt$$

Thus for the action to be an extremum, we need

$$-\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial y'} + \frac{\partial \mathcal{L}}{\partial y} = 0 \tag{2}$$

Using equation (1), we find:

$$\frac{d}{dt} \left(2\mu \dot{y}\right) + \frac{d}{dx} \left(-2Ty'\right) - 0 = 0$$

or

$$\mu \ddot{y} - Ty'' = 0$$

which is the wave equation for the string.

An alternative approach is to write the string displacement as a sum over normal modes:

$$y = \sum_n y_n \left(t\right) \sin \frac{n\pi x}{L}$$

Then the Lagrangian density (1) is

$$\mathcal{L} = \sum_n \sum_p \mu \dot{y}_n \dot{y}_p \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} - T \frac{\pi^2}{L^2} np y_n y_p \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}$$

and then the Lagrangian is

$$L = \int \mathcal{L} dx$$

When we integrate over $x$, the only terms that survive are those with $n = p$

$$L = \frac{L}{2} \sum_n \left( \mu \left(\dot{y}_n\right)^2 - T \frac{\pi^2}{L^2} n^2 y_n^2 \right) \tag{3}$$

The mode amplitudes $y_n$ act as the generalized coordinates for the string. Then Lagrange's equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}_n} - \frac{\partial L}{\partial y_n} = \mu \ddot{y}_n - T \frac{\pi^2}{L^2} n^2 y_n = 0$$

which is the harmonic oscillator equation with frequency $\omega_n = \left(n\pi/L\right) \sqrt{T/\mu}$.

## 2 Lagrangian for the electromagnetic field

Now we want to do a similar treatment for the EM field. We want a Lagrangian density such that the action

$$S = \int \mathcal{L} d^4x$$

is a Lorentz invariant, and where $\mathcal{L}$ is a function of the fields. The "obvious" invariant to try is

$$\mathcal{L}_{\text{guess}} = F^{\alpha\beta} F_{\alpha\beta}$$

(Recall this is proportional to $E^2 - B^2$, an "energy-like" thing.) Here the components of the potential $A^\alpha$ are the "normal modes" – they behave like the $y_n$ in the previous section. Then Lagrange's equations (2) are:

$$\frac{d}{dx_\mu} \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial A^\alpha}{\partial x_\mu} \right)} - \frac{\partial \mathcal{L}}{\partial A^\alpha} = 0 \tag{4}$$

To evaluate this, note that

$$\mathcal{L}_{\text{guess}} = \left( \partial^\alpha A^\beta - \partial^\beta A^\alpha \right) g_{\alpha\gamma} g_{\beta\delta} \left( \partial^\gamma A^\delta - \partial^\delta A^\gamma \right)$$

and so

$$\begin{aligned}
\frac{\partial \mathcal{L}_{\text{guess}}}{\partial \left( \partial^\mu A^\nu \right)} &= \left( \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu \right) g_{\alpha\gamma} g_{\beta\delta} \left( \partial^\gamma A^\delta - \partial^\delta A^\gamma \right) + \left( \partial^\alpha A^\beta - \partial^\beta A^\alpha \right) g_{\alpha\gamma} g_{\beta\delta} \left( \delta^\gamma_\mu \delta^\delta_\nu - \delta^\delta_\mu \delta^\gamma_\nu \right) \\
&= \left( g_{\mu\gamma} g_{\nu\delta} - g_{\nu\gamma} g_{\mu\delta} \right) \left( \partial^\gamma A^\delta - \partial^\delta A^\gamma \right) + \left( \partial^\alpha A^\beta - \partial^\beta A^\alpha \right) \left( g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \right) \\
&= \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) - \left( \partial_\nu A_\mu - \partial_\mu A_\nu \right) + \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) - \left( \partial_\nu A_\mu - \partial_\mu A_\nu \right) \\
&= 4 \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) = 4 F_{\mu\nu}
\end{aligned}$$

while

$$\frac{\partial \mathcal{L}_{\text{guess}}}{\partial A^\nu} = 0$$

So equations (4) become

$$\partial^\mu F_{\mu\nu} = 0$$

which are Maxwell's equations in the absence of sources. We can fix up the Lagrangian by adding the interaction term $\frac{1}{c} J_\alpha A^\alpha$. Thus

$$\mathcal{L} = -\frac{1}{16\pi} F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

With this Lagrangian density

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha$$

and Lagrange's equations become

$$-\frac{1}{4\pi} \partial^\mu F_{\mu\alpha} + \frac{1}{c} J_\alpha = 0$$

or

$$\partial^\mu F_{\mu\alpha} = \frac{4\pi}{c} J_\alpha$$

which are the two Maxwell equations that include sources.

# 3 The Hamiltonian

Now we form the Hamiltonian. First let's look at the string. Using equation (3):

$$\begin{aligned}
H &= \sum_n \frac{\partial \mathcal{L}}{\partial \dot{y}_n} \dot{y}_n - \mathcal{L} \\
&= \frac{L}{2} \sum_n 2\mu \left(\dot{y}_n\right)^2 - \left(\mu \left(\dot{y}_n\right)^2 - T\frac{\pi^2}{L^2} n^2 y_n^2\right) \\
&= \frac{L}{2} \sum_n 2\mu \left(\dot{y}_n\right)^2 + T\frac{\pi^2}{L^2} n^2 y_n^2 = \sum_n E_n
\end{aligned}$$

where $E_n$ is the total (kinetic plus potential) energy per mode. By analogy, we get for the EM field system without sources

$$\begin{aligned}
T^{\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha A_\mu\right)} \partial^\beta A_\mu - g^{\alpha\beta} \mathcal{L} \\
&= -\frac{1}{4\pi} F^{\mu\alpha} \partial^\beta A_\mu - g^{\alpha\beta} \left(-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}\right) \\
&= \frac{1}{4\pi} \left(F^{\alpha\mu} \partial^\beta A_\mu + \frac{1}{4} g^{\alpha\beta} F^{\mu\nu} F_{\mu\nu}\right)
\end{aligned}$$

This tensor is not symmetric, because the first term contains only one half of the field tensor: $\partial^\beta A_\mu$rather than $F^\beta{}_\mu$The conservation laws require that the energy tensor be symmetric, so we have to modify the result.

# 4 The energy-momentum tensor

Recall that the field energy density (non-relativistic) is $\frac{1}{8\pi}\left(E^2 + B^2\right)$and the Poynting theorem may be written

$$\frac{\partial}{\partial t} \frac{1}{8\pi} \left(E^2 + B^2\right) + \vec{\nabla} \cdot \frac{c}{4\pi} \vec{E} \times \vec{B} + \vec{j} \cdot \vec{E} = 0 \tag{5}$$

We'd like to express this result in covariant form. We obviously need something quadratic in the fields. For example:

$$
\begin{aligned}
F^{\alpha}_{\ \mu}F^{\mu\beta} &= \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} E_x^2+E_y^2+E_z^2 & E_yB_z-E_zB_y & E_zB_x-E_xB_z & E_xB_y-E_yB_x \\ E_yB_z-E_zB_y & -E_x^2+B_y^2+B_z^2 & -E_xE_y-B_xB_y & -E_xE_z-B_xB_z \\ E_zB_x-E_xB_z & -E_xE_y-B_xB_y & -E_y^2+B_x^2+B_z^2 & -E_yE_z-B_yB_z \\ E_xB_y-E_yB_x & -E_xE_z-B_xB_z & -E_yE_z-B_yB_z & -E_z^2+B_x^2+B_y^2 \end{pmatrix} \\
&= \begin{pmatrix} E^2 & \left(\vec{E}\times\vec{B}\right)_x & \left(\vec{E}\times\vec{B}\right)_y & \left(\vec{E}\times\vec{B}\right)_z \\ \left(\vec{E}\times\vec{B}\right)_x & -E_x^2+B_y^2+B_z^2 & -E_xE_y-B_xB_y & -E_xE_z-B_xB_z \\ \left(\vec{E}\times\vec{B}\right)_y & -E_xE_y-B_xB_y & -E_y^2+B_x^2+B_z^2 & -E_yE_z-B_yB_z \\ \left(\vec{E}\times\vec{B}\right)_z & -E_xE_z-B_xB_z & -E_yE_z-B_yB_z & -E_z^2+B_x^2+B_y^2 \end{pmatrix}
\end{aligned}
$$

Now we'd like the (0,0) component to be the energy density. We can get that if we add the tensor $\frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}\left(B^2-E^2\right)$. Then

$$
\begin{aligned}
\Theta^{\alpha\beta} &= \frac{1}{4\pi}\left\{F^{\alpha}_{\ \mu}F^{\mu\beta}+\frac{1}{4}g^{\alpha\beta}F^{\mu\varpi}F_{\mu\nu}\right\} \\
&= \frac{1}{4\pi}\begin{pmatrix} E^2+\frac{B^2-E^2}{2} & \left(\vec{E}\times\vec{B}\right)_x & \left(\vec{E}\times\vec{B}\right)_y & \left(\vec{E}\times\vec{B}\right)_z \\ \left(\vec{E}\times\vec{B}\right)_x & -E_x^2+B_y^2+B_z^2-\frac{B^2-E^2}{2} & -E_xE_y-B_xB_y & -E_xE_z-B_xB_z \\ \left(\vec{E}\times\vec{B}\right)_y & -E_xE_y-B_xB_y & -E_y^2+B_x^2+B_z^2-\frac{B^2-E^2}{2} & -E_yE_z-B_yB_z \\ \left(\vec{E}\times\vec{B}\right)_z & -E_xE_z-B_xB_z & -E_yE_z-B_yB_z & -E_z^2+B_x^2+B_y^2-\frac{B^2-E^2}{2} \end{pmatrix} \\
&= \frac{1}{4\pi}\begin{pmatrix} \frac{B^2+E^2}{2} & \left(\vec{E}\times\vec{B}\right)_x & \left(\vec{E}\times\vec{B}\right)_y & \left(\vec{E}\times\vec{B}\right)_z \\ \left(\vec{E}\times\vec{B}\right)_x & \frac{B^2+E^2}{2}-E_x^2-B_x^2 & -E_xE_y-B_xB_y & -E_xE_z-B_xB_z \\ \left(\vec{E}\times\vec{B}\right)_y & -E_xE_y-B_xB_y & \frac{B^2+E^2}{2}-E_y^2-B_y^2 & -E_yE_z-B_yB_z \\ \left(\vec{E}\times\vec{B}\right)_z & -E_xE_z-B_xB_z & -E_yE_z-B_yB_z & \frac{B^2+E^2}{2}-E_z^2-B_z^2 \end{pmatrix}
\end{aligned}
$$

Then

$$\partial_\alpha\Theta^{\alpha 0}=\frac{\partial}{\partial ct}\frac{B^2+E^2}{8\pi}+\vec{\nabla}\left(\frac{\vec{E}\times\vec{B}}{4\pi}\right)$$

which is part of equation (5). On the right hand side we need $\frac{1}{c}\vec{j}\cdot\vec{E}=\frac{1}{c}J_\alpha F^{\alpha 0}$. Thus we have the relation

$$\partial_\alpha\Theta^{\alpha 0}=\frac{1}{c}J_\alpha F^{\alpha 0}$$

and so we guess that the full set of conservation laws are given by:

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{c} J_\alpha F^{\alpha\beta}$$

I leave it to you to show that the $\beta = i$ components give momentum conservation (Jackson equation 6.122).

# 5 Angular momentum

Cross products are not proper vectors. They are pseudo-vectors because they do not transform properly under reflections. Thus it is usually better to express quantities such as angular momentum of a particle ($\vec{L} = r \times \vec{p}$) as antisymmetric tensors. For example the tensor

$$M_{ik} = \sum_{\text{particles}} (x_i p_k - x_k p_i)$$

has three independent compomemts: the components of the vector $\vec{L}$.

Extending this idea, let's look at the tensor

$$M^{\alpha\beta} = \sum_{\text{particles}} \left( x^\alpha p^\beta - x^\beta p^\alpha \right)$$

The 3×3 spacelike part is the tensor $M_{ik}$ and thus represents the angular momentum of the system. In addition:

$$M^{i0} = \sum \left( x^i \frac{\varepsilon}{c} - ctp^i \right)$$

where $\varepsilon$ is the energy of the particle. Conservation of angular momentum for the system is expressed as $M_{ij} =$ constant. Thus we conjecture that the full conservation law is $M^{\alpha\beta} =$ constant. (Or equivalently $\partial_\alpha M^{\alpha\beta} = 0$) This gives for the $(i, 0)$ component:

$$M^{i0} = \sum \left( x^i \frac{\varepsilon}{c} - ctp^i \right) = \text{constant}$$

Now if we divide through by $\sum \varepsilon$ we get

$$\frac{\sum x^i \varepsilon}{\sum \varepsilon} = c^2 t \frac{\sum p^i}{\sum \varepsilon}$$

The term on the left hand side is the position of the center of mass,

$$\vec{r}_{CM} = \frac{\sum \gamma m \vec{x}}{\sum \gamma m}$$

while the term on the right hand side is the CM velocity times $t$.

$$\vec{v}_{CM} = \frac{\sum \gamma m \vec{v}}{\sum \gamma m}$$

an eminently sensible result.

To get the equivalent result for the EM field we form the tensor

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^{\gamma} - \Theta^{\alpha\gamma} x^{\beta}$$

and then the conservation laws should be given by:

$$\partial_{\alpha} M^{\alpha\beta\gamma} = 0$$

Taking $\beta = 0$ gives the CM motion as above.

# 6 The Darwin Lagrangian

The analysis above is for source-free fields. We might attempt to add the free-particle Lagrangian to get a complete description of the particle-plus-field system, but this approach fails because of retardation effects. (The fields propagate at the speed of light.) We can calculate a complete Lagrangian in a single reference frame, inlcuding relativistic effects up to order $\beta^2 = (v/c)^2$.

Let's start with a 2-particle system. Both particles produce fields and both can move under the influence of those fields. The interaction term for charge 1 interacting with the fields due to 2 is

$$\frac{q_1}{c} u_{1,\alpha} A_2^{\alpha} = \frac{q_1}{c} \left( c\gamma, \gamma \vec{v}_1 \right) \left( \phi_2, \vec{A}_2 \right) = q_1 \gamma \left( \phi_2 - \frac{\vec{v}_1}{c} \cdot \vec{A}_2 \right) \tag{6}$$

If we now work in a single reference frame and use the coordinate time rather than proper time as our time variable, we should drop the factor $\gamma$. We want to evaluate this expression to second order in $v/c$. If we work in Coulomb gauge, the potential $\phi_2 = q_2/r$ is exact. We only need $\vec{A}$ to first order since it appears in combination with $v_1/c$. This means we can ignore retardation effects. Then:

$$\vec{A}_2 \simeq \frac{1}{c} \int \frac{\vec{j}_t}{|\vec{x} - \vec{x}'|} dV'$$

where the transverse current is

$$\begin{aligned} \vec{j}_t &= \vec{j} - \vec{j}_l = q_2 \vec{v}_2 \delta\left(\vec{x} - \vec{x}_2\right) - \frac{1}{4\pi} \vec{\nabla} \int \frac{\vec{\nabla}' \cdot q_2 \vec{v}_2 \delta\left(\vec{x}' - \vec{x}_2\right)}{|\vec{x} - \vec{x}'|} dV' \\ &= q_2 \vec{v}_2 \delta\left(\vec{x} - \vec{x}_2\right) - \frac{q_2}{4\pi} \vec{\nabla} \frac{\vec{v}_2 \cdot \left(\vec{x} - \vec{x}_2\right)}{\left|\vec{x} - \vec{x}_2\right|^3} \end{aligned}$$

Therefore

$$\vec{A} = \frac{q_2}{r} \frac{\vec{v}_2}{c} - \frac{q_2}{4\pi c} \int \frac{1}{|\vec{x} - \vec{x}'|} \vec{\nabla}' \frac{\vec{v}_2 \cdot \left(\vec{x}' - \vec{x}_2\right)}{\left|\vec{x}' - \vec{x}_2\right|^3} dV'$$

Let's look at the integral. First make a change of origin. Let $\vec{u} = \vec{x}' - \vec{x}_2$ and with $\vec{r} = \vec{x} - \vec{x}_2$, we get

$$\begin{aligned}
\int \frac{1}{|\vec{x}-\vec{x}'|} \vec{\nabla}' \frac{\vec{v}_2 \cdot (\vec{x}' - \vec{x}_2)}{|\vec{x}' - \vec{x}_2|^3} dV' &= \int \frac{1}{|\vec{u}-\vec{r}|} \vec{\nabla}_u \frac{\vec{v}_2 \cdot \vec{u}}{u^3} d^3\vec{u} \\
&= \frac{1}{|\vec{u}-\vec{r}|} \frac{\vec{v}_2 \cdot \vec{u}}{u^3}\bigg|_{S \text{ at } \infty} - \int \vec{\nabla}_u \left( \frac{1}{\vec{u}-\vec{r}} \right) \frac{\vec{v}_2 \cdot \vec{u}}{u^3} d^3\vec{u} \\
&= \vec{\nabla}_r \int \frac{1}{|\vec{u}-\vec{r}|} \frac{\vec{v}_2 \cdot \hat{u}}{u^2} d^3\vec{u} \\
&= \vec{\nabla}_r \int \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\mu)\, \vec{v}_2 \cdot \hat{u}\ du d\mu d\phi
\end{aligned}$$

where we have put the polar axis for $\vec{u}$ along $\vec{r}$. Now

$$\vec{v}_2 \cdot \hat{u} = \vec{v}_2 \cdot (\hat{r}\cos\theta + \sin\theta\, (\hat{x}\cos\phi + \hat{y}\sin\phi))$$

Integration over $\phi$ renders the $x-$ and $y-$components zero.

Next we make use of the orthogonality of the $P_l(\mu)$, noting that $\cos\theta = P_1(\mu)$. Only $l = 1$ survives the integration over $\mu$. We obtain:

$$\begin{aligned}
\text{integral} &= 2\pi \vec{\nabla}_r \int_0^\infty \frac{r_<}{r_>^2} \frac{2}{3} \vec{v}_2 \cdot \hat{r}\ du \\
&= \frac{4\pi}{3} \vec{\nabla}_r \vec{v}_2 \cdot \hat{r} \left( \int_0^r \frac{u}{r^2} du + \int_r^\infty \frac{r}{u^2}\ du \right) \\
&= \frac{4\pi}{3} \vec{\nabla}_r \vec{v}_2 \cdot \hat{r} \left( \frac{1}{2} + 1 \right) = 2\pi \vec{\nabla}_r \left( \vec{v}_2 \cdot \frac{\vec{r}}{r} \right)
\end{aligned}$$

And thus

$$\begin{aligned}
\vec{A}_2 &= \frac{q_2}{r} \frac{\vec{v}_2}{c} - \frac{q_2}{4\pi c} 2\pi \vec{\nabla}_r \left( \vec{v}_2 \cdot \frac{\vec{r}}{r} \right) \\
&= \frac{q_2}{c} \left[ \frac{\vec{v}_2}{r} - \frac{1}{2} \left( \frac{\vec{v}_2}{r} - \frac{\vec{v}_2 \cdot \vec{r}}{r^2} \hat{r} \right) \right] \\
&= \frac{q_2}{2cr} \left[ \vec{v}_2 + (\vec{v}_2 \cdot \hat{r})\, \hat{r} \right]
\end{aligned}$$

Then the interaction term for 2 particles (equation 6 with the $\gamma$ dropped) is:

$$\begin{aligned}
q_1 \left( \phi_2 - \frac{\vec{v}_1 \cdot \vec{A}_2}{c} \right) &= q_1 \left( \frac{q_2}{r} - \frac{\vec{v}_1}{c} \cdot \frac{q_2}{2cr} \left[ \vec{v}_2 + (\vec{v}_2 \cdot \hat{r})\, \hat{r} \right] \right) \\
&= q_1 \frac{q_2}{r} \left\{ 1 - \frac{1}{2c^2} \left[ \vec{v}_1 \cdot \vec{v}_2 + (\vec{v}_2 \cdot \hat{r})\, (\vec{v}_1 \cdot \hat{r}) \right] \right\}
\end{aligned}$$

Adding this term to the kinetic energy (Lagrangian notes pg 5), we have the Darwin Lagrangian for a collection of charged particles:

$$L_D = -\frac{1}{2} \sum_i m_i c^2 \sqrt{1 - \frac{v_i^2}{c^2}} - \sum_{i>j} \frac{q_i q_j}{r_{ij}} \left\{ 1 - \frac{1}{2c^2} \left[ \vec{v}_i \cdot \vec{v}_j + (\vec{v}_i \cdot \hat{r})\, (\vec{v}_j \cdot \hat{r}) \right] \right\}$$

To be consistent, we should evaluate the first term to second order in $v/c$,i.e. $\left(1 - v^2/c^2\right)^{1/2} \simeq 1 - \frac{1}{2}\frac{v^2}{c^2} + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{v^4}{2c^4}$. Finally, dropping the constant leading term which is irrelevant, we have

$$L_D = \frac{1}{2}\sum_i m_i v_i^2 + \frac{1}{8c^2}\sum_i m_i v_i^4 - \sum_{i>j}\frac{q_i q_j}{r_{ij}}\left\{1 - \frac{1}{2c^2}\left[\vec{v}_i \cdot \vec{v}_j + \left(\vec{v}_i \cdot \hat{r}\right)\left(\vec{v}_j \cdot \hat{r}\right)\right]\right\}$$

correct to second order in $v/c$.