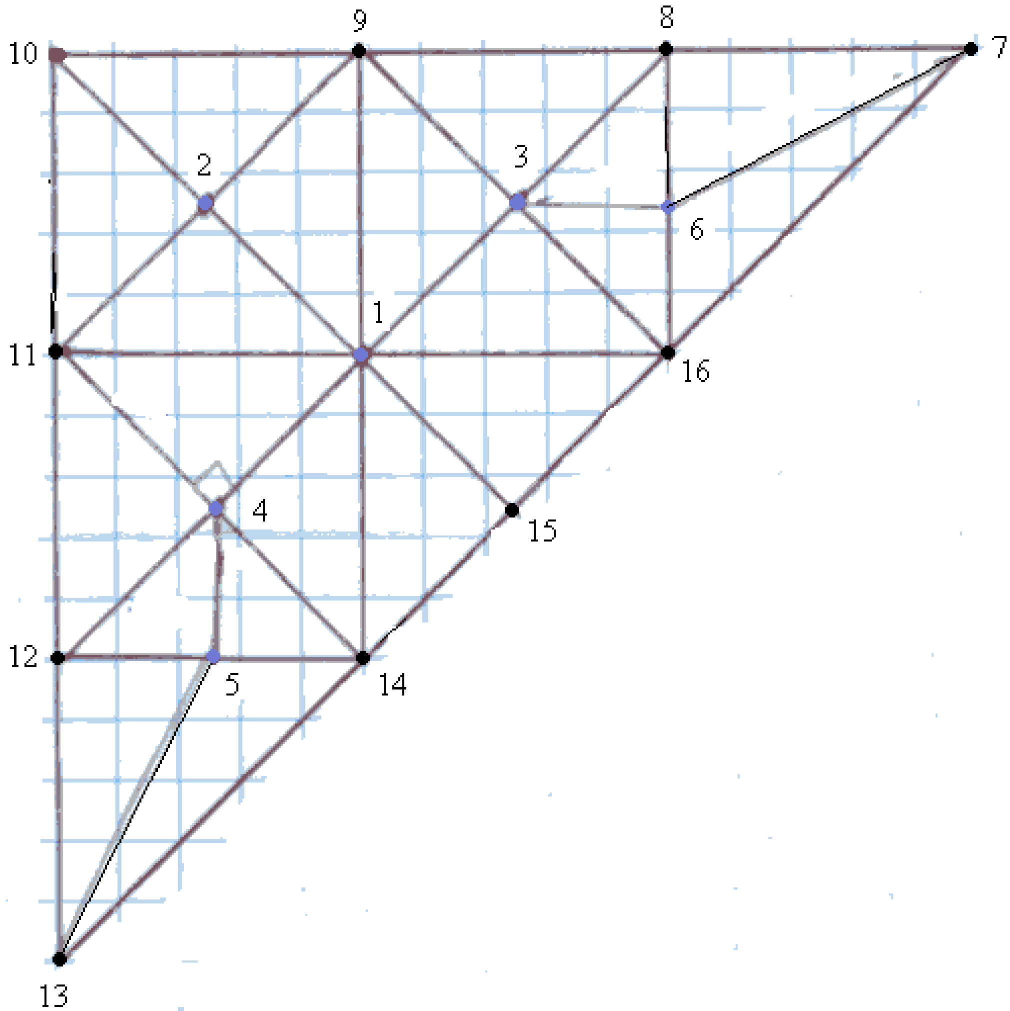
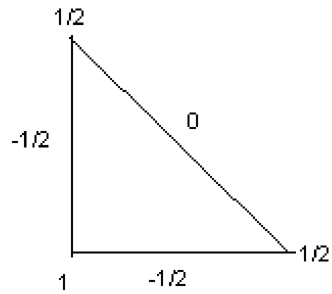
January 2020

Finite element analysis example.

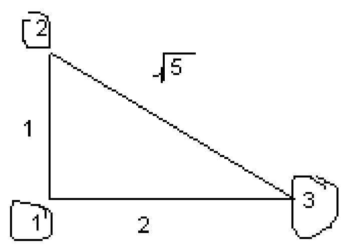
We set up a triangle with 6 internal nodes,as shown.



In this grid we have three different triangular elements. One is the isosceles right triangle treated by Jackson (Fig 2.16)



We also have a right triangle with sides 1,2 and $\sqrt{5}$, and area 1. (Nodes 6,7,8 form this triangle, and also nodes 5,12,13.)



For this triangle we have

$$\begin{aligned} a_1 &= \frac{1}{2A}\left(x_2y_3 - x_3y_2\right) = \frac{1}{2}\left(0-2\right) = -1 \\ b_1 &= \frac{1}{2A}\left(y_2 - y_3\right) = \frac{1}{2}\left(1-0\right) = \frac{1}{2} \\ c_1 &= -\frac{1}{2A}\left(x_2 - x_3\right) = -\frac{1}{2}\left(0-2\right) = 1 \end{aligned}$$

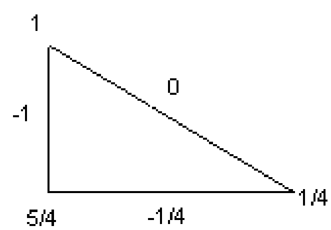
$$\begin{aligned} a_2 &= \frac{1}{2A}\left(x_3y_1 - x_1y_3\right) = \frac{1}{2}\left(0-0\right) = 0 \\ b_2 &= \frac{1}{2A}\left(y_3 - y_1\right) = \frac{1}{2}\left(0-0\right) = 0 \\ c_2 &= -\frac{1}{2A}\left(x_3 - x_1\right) = -\frac{1}{2}\left(2-0\right) = -1 \end{aligned}$$

$$\begin{aligned} a_3 &= \frac{1}{2A}\left(x_1y_2 - x_2y_1\right) = \frac{1}{2}\left(0-0\right) = 0 \\ b_3 &= \frac{1}{2A}\left(y_1 - y_2\right) = \frac{1}{2}\left(0-1\right) = -\frac{1}{2} \\ c_3 &= -\frac{1}{2A}\left(x_1 - x_2\right) = -\frac{1}{2}\left(0-0\right) = 0 \end{aligned}$$

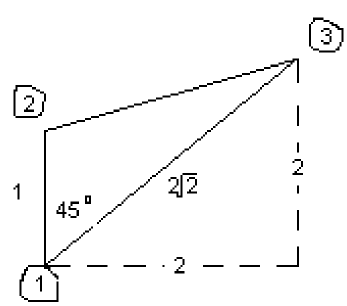
Note: Jackson claims that $\sum a_i = 1$, but the determinant is $\pm 2A$ depending on how the vertices are labelled, which explains why I have -1. (In other words, my $D = -2A$.) But the k_{ij} all depend quadratically on the bs and cs, and so are independent of their signs. For this triangle,

$$k_{11} = A(b_1^2 + c_1^2) = \frac{1}{4} + 1 = \frac{5}{4}$$
$$k_{12} = A\left(b_1 b_2 + c_1 c_2\right) = \left(\frac{1}{2}\right)(0) + 1 \times (-1) = -1$$
$$k_{13} = A\left(b_1 b_3 + c_1 c_3\right) = \frac{1}{2} \times \left(-\frac{1}{2}\right) + 1 \times 0 = -\frac{1}{4}$$
$$k_{22} = A(b_2^2 + c_2^2) = 0 + 1 = 1$$
$$k_{23} = A\left(b_2 b_3 + c_2 c_3\right) = 0 + 0 = 0$$
$$k_{33} = A(b_3^2 + c_3^2) = \frac{1}{4} + 0 = \frac{1}{4}$$

Thus we have:



The third triangle is formed by nodes 5,13,14 and also 6,7,16 and looks like this:



This triangle also has area 1 and its shape functions have coefficients:

$$
\begin{aligned}
a_1 &= \frac{1}{2A}(x_2y_3 - x_3y_2) = \frac{1}{2}(0 - 2\times 1) = -1 \\
b_1 &= \frac{1}{2A}(y_2 - y_3) = \frac{1}{2}(1-2) = -\frac{1}{2} \\
c_1 &= -\frac{1}{2A}(x_2 - x_3) = -\frac{1}{2}(0-2) = 1
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= \frac{1}{2A}(x_2y_1 - x_1y_3) = \frac{1}{2}(0 - 0) = 0 \\
b_2 &= \frac{1}{2A}(y_3 - y_1) = \frac{1}{2}(2-0) = 1 \\
c_2 &= -\frac{1}{2A}(x_3 - x_1) = -\frac{1}{2}(2-0) = -1
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= \frac{1}{2A}(x_1y_2 - x_2y_1) = \frac{1}{2}(0 - 0) = 0 \\
b_3 &= \frac{1}{2A}(y_1 - y_2) = \frac{1}{2}(0-1) = -\frac{1}{2} \\
c_3 &= -\frac{1}{2A}(x_1 - x_2) = -\frac{1}{2}(0-0) = 0
\end{aligned}
$$

And then

$$k_{11} = A(b_1^2 + c_1^2) = \frac{1}{4} + 1 = \frac{5}{4}$$

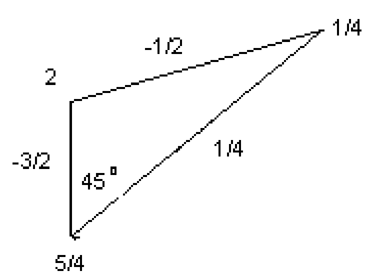
$$k_{12} = A\,(b_1b_2 + c_1c_2) = \left(-\frac{1}{2}\right)(1) + 1\times(-1) = -\frac{3}{2}$$

$$k_{13} = A\,(b_1b_3 + c_1c_3) = \left(-\frac{1}{2}\right)\times\left(-\frac{1}{2}\right) + 1\times 0 = \frac{1}{4}$$

$$k_{22} = A(b_2^2 + c_2^2) = 1 + 1 = 2$$

$$k_{23} = A\,(b_2b_3 + c_2c_3) = 1\left(-\frac{1}{2}\right) + 0 = -\frac{1}{2}$$

$$k_{33} = A(b_3^2 + c_3^2) = \frac{1}{4} + 0 = \frac{1}{4}$$



Now we have to compute the matrix coefficients for the six internal nodes. Eight identical triangles meet at node 1. This node is a vertex of the isosceles right triangle, so

$$K_{11} = 8\left(\frac{1}{2}\right) = 4$$

Similarly four identical triangles meet at node 2. This node is the vertex at the right angle of the isosceles right triangle, so we have

$$K_{22} = 4\,(1) = 4 = K_{33}$$

Four triangles meet at nodes 5 and 6, but only two of the vertices are the same (the vertex at the right angle of the isosceles right triangle). The other two are nodes 2 of the second and third triangles. Thus

$$K_{55} = K_{66} = 2 \times 1 + 2 + 1 = 5$$

Line 12 is a side of two identical triangles, (a side of the isosceles right triangle), so

$$K_{12} = 2\left(-\frac{1}{2}\right) = -1 = K_{13} = K_{14}$$

Similarly

$$K_{45} = K_{36} = 2\left(-\frac{1}{2}\right) = -1$$

and these are the only non-zero elements. Thus

$$K = \begin{pmatrix} 4 & -1 & -1 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 5 & 0 \\ 0 & 0 & -1 & 0 & 0 & 5 \end{pmatrix},$$

Notice that this is a sparse matrix with inverse:

$$K^{-1} = \begin{pmatrix} \frac{76}{245} & \frac{19}{245} & \frac{4}{49} & \frac{4}{49} & \frac{4}{245} & \frac{4}{245} \\ \frac{19}{245} & \frac{66}{245} & \frac{1}{49} & \frac{1}{49} & \frac{1}{245} & \frac{1}{245} \\ \frac{4}{49} & \frac{1}{49} & \frac{265}{931} & \frac{20}{931} & \frac{4}{931} & \frac{53}{931} \\ \frac{4}{49} & \frac{1}{49} & \frac{20}{931} & \frac{265}{931} & \frac{53}{931} & \frac{4}{931} \\ \frac{4}{245} & \frac{1}{245} & \frac{4}{931} & \frac{53}{931} & \frac{984}{4655} & \frac{4}{4655} \\ \frac{4}{245} & \frac{1}{245} & \frac{53}{931} & \frac{4}{931} & \frac{4}{4655} & \frac{984}{4655} \end{pmatrix}$$

or $K^{-1} =$

$$\begin{pmatrix} 0.3102 & 7.755 \times 10^{-2} & 8.163 \times 10^{-2} & 8.163 \times 10^{-2} & 1.633 \times 10^{-2} & 1.633 \times 10^{-2} \\ 7.755 \times 10^{-2} & 0.26939 & 2.041 \times 10^{-2} & 2.041 \times 10^{-2} & 4.082 \times 10^{-3} & 4.082 \times 10^{-3} \\ 8.163 \times 10^{-2} & 2.041 \times 10^{-2} & 0.2846 & 2.148 \times 10^{-2} & 4.296 \times 10^{-3} & 5.693 \times 10^{-2} \\ 8.163 \times 10^{-2} & 2.041 \times 10^{-2} & 2.148 \times 10^{-2} & 0.28464 & 5.693 \times 10^{-2} & 4.296 \times 10^{-3} \\ 1.633 \times 10^{-2} & 4.082 \times 10^{-3} & 4.296 \times 10^{-3} & 5.693 \times 10^{-2} & 0.21139 & 8.593 \times 10^{-4} \\ 1.633 \times 10^{-2} & 4.082 \times 10^{-3} & 5.693 \times 10^{-2} & 4.296 \times 10^{-3} & 8.593 \times 10^{-4} & 0.21139 \end{pmatrix}$$

Now we set up boundary conditions with the sloping side (nodes 7, 13, 14, 15, 16) grounded and the other two sides at unit potential. Then the source vector includes the nodes on the boundaries, which are labelled 7 through 16 (finelemnotes eqn 9):

$$-G_1 = \sum_{j=7}^{16} K_{1j}\psi_j = K_{1,9}\psi_9 + K_{1,11}\psi_{11} + K_{1,14}\psi_{14} + K_{1,15}\psi_{15} + K_{1,16}\psi_{16}$$

The only nonzero K in this list is $K_{1,15}$ but $\psi_{15} = 0$, so $G_1 = 0$. Similarly:

$$\begin{aligned} -G_2 &= K_{2,9}\psi_9 + K_{2,10}\psi_{10} + K_{2,11}\psi_{11} \\ &= 3 \times 2(-\frac{1}{2}) \times 1 = -3 \end{aligned}$$

$$\begin{aligned} -G_3 &= K_{3,9}\psi_9 + K_{38}\psi_8 + K_{3,16}\psi_{16} \\ &= 2(-\frac{1}{2}) \times 1 + \left(-\frac{1}{2} + 0\right) \times 1 + \left(-\frac{1}{2} + 0\right) \times 0 \\ &= -1 - \frac{1}{2} = -\frac{3}{2} = -G_4 \end{aligned}$$

$$\begin{aligned} -G_5 &= K_{5,12}\psi_{12} + K_{5,13}\psi_{13} + K_{5,14}\psi_{14} \\ &= \left(-\frac{1}{2} - 1\right) \times 1 + 0 + 0 \\ &= -\frac{3}{2} = -G_6 \end{aligned}$$

Then to solve the problem, we multiply

$$\Phi = K^{-1}G$$

$$\begin{pmatrix} \frac{76}{245} & \frac{19}{245} & \frac{4}{49} & \frac{4}{49} & \frac{4}{245} & \frac{4}{245} \\ \frac{19}{245} & \frac{66}{245} & \frac{1}{49} & \frac{1}{49} & \frac{1}{245} & \frac{1}{245} \\ \frac{4}{49} & \frac{1}{49} & \frac{265}{931} & \frac{20}{931} & \frac{4}{931} & \frac{53}{931} \\ \frac{4}{49} & \frac{1}{49} & \frac{20}{931} & \frac{265}{931} & \frac{53}{931} & \frac{4}{931} \\ \frac{4}{245} & \frac{1}{245} & \frac{4}{931} & \frac{53}{931} & \frac{984}{4655} & \frac{4}{4655} \\ \frac{4}{245} & \frac{1}{245} & \frac{53}{931} & \frac{4}{931} & \frac{4}{4655} & \frac{984}{4655} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 0.526\,53 \\ 0.881\,63 \\ 0.612\,24 \\ 0.612\,24 \\ 0.422\,45 \\ 0.422\,45 \end{pmatrix}$$

The greatest value is at node 2, which is closest to the sides at unit potential. The potential is equal at nodes 3 and 4, and also equal at nodes 5 and 6. This ii due to ghe reflection symmetry of the system about the line 10-2-1-15. All values are between 0 and 1, as expected.