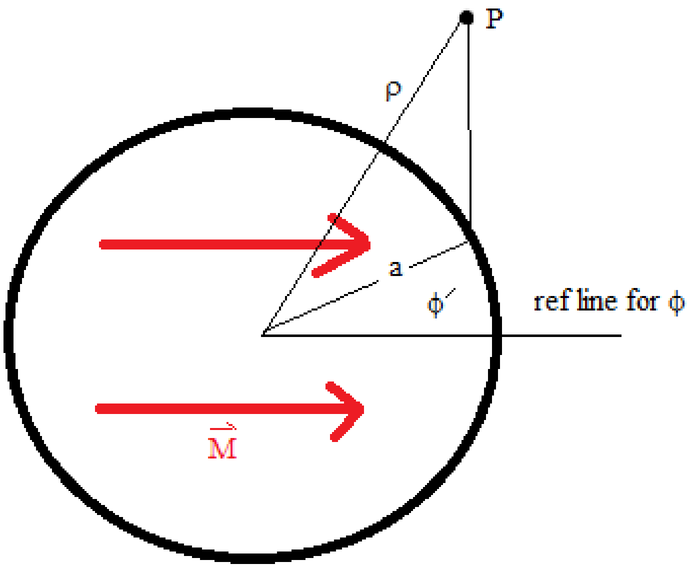
A disk magnet of radius a and height h, $h \ll a$, has uniform magnetizaton $\vec{M}$ throughout its interior. Find the magnetic fields $\vec{B}$ and $\vec{H}$.



We place our coordinate axes with the reference line for ϕ along $\vec{M}$, so that $\vec{M} = M\hat{x}$. As we have seen, (notes 1 equation 50) there is an effective magnetic "surface charge density" $\vec{M} \cdot \hat{n}$ at the surface of the magnet. In this case we have

$$\sigma_M = \vec{M} \cdot \hat{n} = M\hat{x} \cdot \hat{\rho} = M \cos \phi$$

at $\rho = a$. The magnetic scalar potential is then given by (eqn 51, notes 1)

$$\begin{aligned} \Phi_m(\rho, \phi, z) &= \frac{1}{4\pi} \int_0^h dz' \int_0^{2\pi} \frac{M \cos \phi'}{|\vec{x} - \vec{x}'|} a d\phi' \\ &= \frac{1}{4\pi} \int_0^h dz' \int_0^{2\pi} \frac{M \cos \phi'}{\sqrt{(z - z')^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \phi')}} a d\phi' \end{aligned} \tag{1}$$

The integral is nasty in the general case. Later in the semester we shall develop tools that will allow us to find $\vec{H}$ everywhere. For now we will look at two special cases.

Outside the magnet at a great distance away, $z \gg h \geq z'$, and/or $\rho \gg a > h$, we have

$$
\begin{aligned}
\Phi_m &= \frac{1}{4\pi}\int_0^h \int_0^{2\pi} \frac{M\cos\phi'}{\sqrt{(z-z')^2+\rho^2-2\rho a\cos(\phi-\phi')}} a d\phi' dz' \\
&= \frac{1}{4\pi}\int_0^h \int_0^{2\pi} \frac{M\cos\phi'}{R\sqrt{1-2\frac{\rho a}{R^2}\cos(\phi-\phi')}} a d\phi' dz' \quad \text{where } R^2 \equiv (z-z')^2+\rho^2 \\
&= \frac{1}{4\pi}\int_0^h \int_0^{2\pi} \frac{\cos\phi'}{R}\left\{1+\frac{\rho a}{R^2}\cos(\phi-\phi')+\cdots\right\} a\ d\phi' dz'
\end{aligned}
$$

Since either z or $\rho \gg h > z'$,

$$R^2 \simeq z^2+\rho^2$$

which is independent of z'. Thus the z' integration results in a multiplication by h. Doing the ϕ' integration, the first term integrates to zero. We expand the cosine in the second term, and make use of the orthogonality of the trig functions to get:

$$
\begin{aligned}
\Phi_m &= \frac{h}{4\pi}\left[\cos\phi'\frac{\rho a}{R^2}\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right)+\cdots\right] a d\phi' \\
&= \frac{Mh}{4\pi R^3}\pi a^2\rho\cos\phi = \frac{Mh\pi a^2}{4\pi}\frac{x}{(z^2+\rho^2)^{3/2}}
\end{aligned}
$$

With the magnetic moment $\vec{m}$ equal to $\vec{M}V = \pi a^2 hM\hat{x}$, the potential is

$$\Phi_m = \frac{\vec{m}\cdot\vec{r}}{4\pi R^3}$$

as expected for a dipole. The result is valid for $R \gg a$. (Compare with eqns 33 and 49 in Notes 1). In this region $\vec{B} = \mu_0\vec{H}$ because $\vec{M}$ is zero outside the magnet.

Inside the magnet and near the axis, $z-z' \ll a$, $\rho \ll a$, we may expand eqn (1) in a different way. Letting $R^2 = (z-z')^2+\rho^2$, and keeping terms up to the square of ρ/a and R/a, we have

$$
\begin{aligned}
\Phi_m = \frac{1}{4\pi}\int_0^h dz' \int_0^{2\pi} \frac{M\cos\phi'}{a}\Bigg\{1-\frac{1}{2}\frac{R^2-2\rho a\cos(\phi-\phi')}{a^2} \\
+\frac{3}{8}\left[2\frac{\rho}{a}\cos(\phi-\phi')\right]^2+\cdots\Bigg\} a\ d\phi' dz'
\end{aligned}
$$

The first two terms give zero when we integate over ϕ', and the remainder are independent of z', leaving

$$
\begin{aligned}
\Phi_m &\simeq \frac{hM}{4\pi}\int_0^{2\pi}\cos\phi'\left\{\frac{\rho}{a}\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right)\right.\\
&\qquad\left.+\frac{3}{2}\left(\frac{\rho}{a}\right)^2\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right)^2\cdots\right\}d\phi'\\
&= \frac{hM}{4a}x+0+\frac{hM}{4\pi}\frac{3}{2}\left(\frac{\rho}{a}\right)^2\int_0^{2\pi}\cos\phi'\left[\cos^2\phi\frac{\left(1+\cos 2\phi'\right)}{2}+\frac{\sin 2\phi\sin 2\phi'}{2}\right.\\
&\qquad\left.+\sin^2\phi\frac{\left(1-\cos 2\phi'\right)}{2}\right]\,d\phi'+\cdots\\
&= \frac{hM}{4a}x
\end{aligned}
$$

where again we used the orthogonality of the trig functions. The linear potential gives a uniform field:

$$\vec{H}=-\vec{\nabla}\Phi_m=-\frac{hM}{4a}\hat{x}=-\frac{h}{4a}\vec{M}$$

Note that $\vec{H}$ is opposite $\vec{M}$. The magnetic induction $\vec{B}$ is

$$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)=\mu_0\vec{M}\left(1-\frac{h}{4a}\right)$$

and is in the same direction as $\vec{M}$.

Because $\vec{\nabla}\cdot\vec{B}=0$, the lines of $\vec{B}$ form closed loops. On the other hand, $\vec{H}$ diverges from the positive and negative sources on the two sides of the magnet. Thus $\vec{H}$ is opposite $\vec{B}$ in the interior. Although we obtained this result in a special case, it is quite general.