Diffraction

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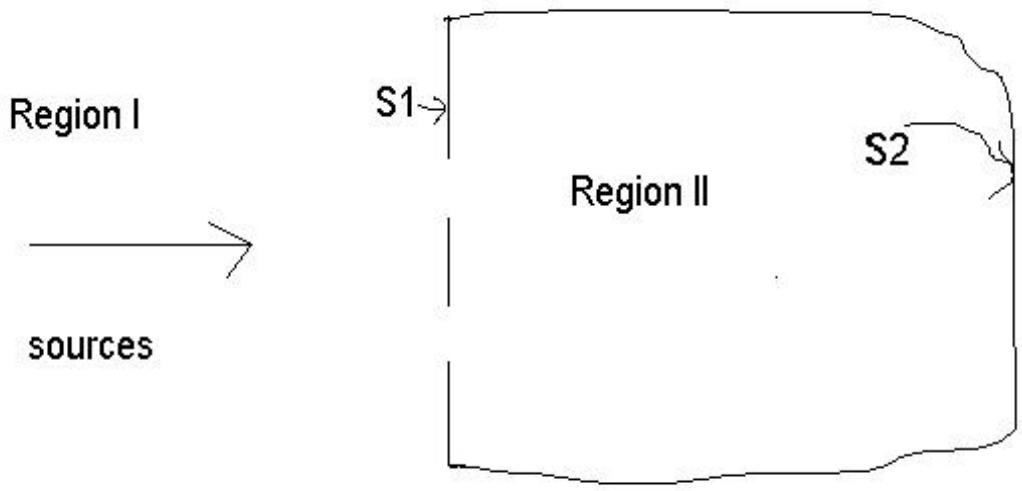
Diffraction occurs when EM waves approach an aperture (or an obstacle) with dimension $d > \lambda$. We shall refer to the region containing the source of the waves as region I and the region containing the diffracted fields as region II. The total volume comprises both regions I and II. Region II is bounded by a surface $S = S_1 + S_2$. The surface S_2 is at "infinity", while S_1 is a screen containing apertures, or else a set of obstacles.



1 Kirchhoff's method

We may write all the fields in the form

$$\psi \propto e^{-i\omega t}$$

and then ψ satisfies the Helmholtz equation

$$\left(\nabla^2 + k^2\right)\psi = 0$$

where $k = \omega/c$ and ψ is a component of $\vec{E}$ or $\vec{B}$, or another variable describing the field, such as a component of $\vec{A}$. We can solve the Helmholtz equation with the outgoing wave Green's function. Since there are no sources in region II, the volume integral is zero.

$$\psi(\vec{x}) = \int_S \left[\psi(\vec{x}')\,\hat{\mathbf{n}}\cdot\vec{\nabla}'G - G\hat{\mathbf{n}}\cdot\vec{\nabla}'\psi(\vec{x}')\right] da' \tag{1}$$

where

$$G = \frac{1}{4\pi}\frac{e^{ikR}}{R}$$

Thus

$$\psi(\vec{x}) = -\frac{1}{4\pi}\int_{S_1+S_2}\frac{e^{ikR}}{R}\left[\psi(\vec{x}')\,ik\left(1+\frac{i}{kR}\right)\hat{\mathbf{n}}\cdot\hat{\mathbf{R}} + \hat{\mathbf{n}}\cdot\vec{\nabla}'\psi(\vec{x}')\right] da' \tag{2}$$

The integral reduces to an integral over S_1, since $\psi \sim 1/R$ at ∞.

We can evaluate the integral using the **Kirchhoff approximation:**

1. ψ and $\frac{\partial\psi}{\partial n}$ vanish on S_1 except in the openings, and
2. in the openings, ψ and $\frac{\partial\psi}{\partial n}$ are equal to the values in the incident field.

Strictly speaking, this is not a mathematically sound procedure, since if *both* ψ and $\frac{\partial\psi}{\partial n}$ vanish on any finite surface, then the solution ψ must be zero. Also the "solution" we obtain does not yield the assumed values of ψ on S_1. Yet the solution does a pretty good job of reproducing experimental results.

We can partially fix things up by using a better Green's function. If ψ is known (or approximated) on S_1, we should use a Dirichlet Green's function, for which $G_D(\vec{x},\vec{x}') = 0$ for $\vec{x}'$ on S_1. Then the solution is:

$$\psi(\vec{x}) = \int_{S_1}\psi(\vec{x}')\,\hat{\mathbf{n}}\cdot\vec{\nabla}'G_D\,da' \tag{3}$$

Similarly, if $\frac{\partial\psi}{\partial n}$ is known or approximated on S_1, we should use the Neumann Green's function, $\frac{\partial G_N}{\partial n} = 0$ on S_1. Then

$$\psi(\vec{x}) = -\int_{S_1} G\hat{\mathbf{n}}\cdot\vec{\nabla}'\psi(\vec{x}')\,da' \tag{4}$$

If the screen S_1 is plane we can easily find the appropriate Green's functions using the method of images:

$$G_{D,N} = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} \mp \frac{e^{ikR'}}{R'}\right)$$

where $\vec{R} = \vec{x} - \vec{x}'$ and $\vec{R}' = \vec{x} - \vec{x}''$ and $\vec{x}''$ describes the point that is the mirror image of $\vec{x}'$ in S_1, and we let both points approach the surface S_1. Then

$$G_D \to 0 \text{ as } z' \to 0$$

and

$$\begin{aligned}\frac{\partial G_N}{\partial n'} &= \frac{\partial G_N}{\partial z'} = \frac{1}{4\pi}\left[\frac{-(z-z')}{R} + \frac{(z+z')}{R}\right]\left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{R} \\ &\to \frac{2z'}{4\pi R}\frac{e^{ikR}}{R}\left(\frac{1}{R} - ik\right) \text{ as } z' \to 0\end{aligned}$$

and thus equation (3) becomes:

$$\begin{aligned}\psi(\vec{x}) &= \int_{S_1} \psi(\vec{x}') \frac{2z}{4\pi R} \frac{e^{ikR}}{R} \left(\frac{1}{R} - ik\right) da' \\ &= \frac{1}{2\pi} \int_{S_1} \psi(\vec{x}') \frac{z}{R} \frac{e^{ikR}}{R} \left(\frac{1}{R} - ik\right) da' \end{aligned} \quad (5)$$

with $z/R = \cos\theta$, while equation (4) becomes:

$$\psi(\vec{x}) = -\frac{1}{2\pi} \int_{S_1} \frac{e^{ikR}}{R} \frac{\partial}{\partial z'} \psi(\vec{x}')\, da' \quad (6)$$

Now to estimate the differences between these expressions, we choose a point source in region I with coordinate $\vec{r}_0$ so that the wave from the source travels in direction $-\hat{\mathbf{r}}_0$ to reach the center of a small aperture at the origin. Then we can write

$$\psi_{\text{inc}} = \psi_0 \frac{e^{i\vec{k}\cdot\vec{R}}}{R} = \psi_0 \frac{e^{-ik\left(r_0 + \hat{\mathbf{r}}_0 \cdot \vec{x}'\right)}}{r_0} = \psi_0 \frac{e^{-ikr_0} e^{-ik\hat{\mathbf{r}}_0 \cdot \vec{x}'}}{r_0}$$

and

$$\frac{\partial \psi_{\text{inc}}}{\partial n} = \psi_0 \frac{e^{-ikr_0} e^{-ik\hat{\mathbf{r}}_0 \cdot \vec{x}'}}{r_0} \left(-ik\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{n}}\right)$$

where $\vec{r}_0 \cdot \hat{\mathbf{n}} = \cos(\pi - \theta_0) = -\cos\theta_0$. Then the 3 expressions for ψ may be written:

outgoing wave Greens function:

$$\begin{aligned}\psi(\vec{x}) &= -\frac{1}{4\pi} \int_{S_1+S_2} \frac{e^{ikR}}{R} \left[\psi(\vec{x}')\, ik \left(1 + \frac{i}{kR}\right) \hat{\mathbf{n}} \cdot \hat{\mathbf{R}} + \frac{\partial}{\partial z'} \psi(\vec{x}')\right] da' \\ &\simeq -\frac{\psi_0}{4\pi} \int_{S_1+S_2} \frac{e^{ikr}}{r} \frac{e^{-ikr_0} e^{-ik\hat{\mathbf{r}}_0 \cdot \vec{x}'}}{r_0} \left[ik\hat{\mathbf{n}} \cdot \hat{\mathbf{R}} + \left(-ik\vec{r}_0 \cdot \hat{\mathbf{n}}\right)\right] da' \\ &\simeq -ik\frac{\psi_0}{4\pi} \int_{S_1+S_2} \frac{e^{ikr}}{r} \frac{e^{-ikr_0}}{r_0} \left[\cos\theta_0 + \cos\theta\right] da' \end{aligned}$$

Dirichlet Green's function:

$$\psi(\vec{x}) = -\frac{ik}{2\pi} \psi_0 \int_{S_1} \frac{e^{ikr}}{r} \frac{e^{-ikr_0}}{r_0} \cos\theta da'$$

and

Neumann Green's function:

$$\psi(\vec{x}) = -\frac{ik}{2\pi} \psi_0 \int_{S_1} \frac{e^{ikr}}{r} \frac{e^{-ikr_0}}{r_0} \cos\theta_0 da'$$

If the aperture is small compared with the distances r and r_0, then the factors cosθ and $\cos\theta_0$ are essentially constant throughout the range of integration. Then all 3 expressions give the same diffraction pattern: the overall intensity smplitude differs by small factors of order 1. Thus we can use whichever of the expressions is most convenient.

2 Fraunhoffer and Fresnel diffraction

Thee exponential that appears in equation (2) may be approximated:

$$\begin{aligned} kR &= k\left|\vec{x}-\vec{x}'\right| = k\sqrt{r^2+r'^2-2\vec{x}\cdot\vec{x}'} \\ &= kr\sqrt{1+\frac{r'^2}{r^2}-2\frac{\hat{\mathbf{r}}\cdot\vec{x}'}{r}} \\ &= kr-k\hat{\mathbf{r}}\cdot\vec{x}'-\frac{k}{2r}\left(r'^2-\left(\hat{\mathbf{r}}\cdot\vec{x}'\right)^2\right)+\cdots \end{aligned}$$

Thus

$$e^{ikR}=e^{ikr}e^{-ik\hat{\mathbf{r}}\cdot\vec{x}'}\exp\left(-i\frac{k}{2r}\left(r'^2-\left(\hat{\mathbf{r}}\cdot\vec{x}'\right)^2\right)\right)$$

The the exponent in the second term is of order:

$$k\hat{\mathbf{r}}\cdot\vec{x}'=\frac{d}{\lambda}$$

where d is the dimension of the aperture, and its ratio to the first term is of order $d/r.$. The exponent in the third term is of order

$$\frac{d^2}{\lambda r}$$

In Fraunhauffer diffraction we ignore the third term. This means that we need

$$\frac{d^2}{\lambda r}\ll 1$$

and

$$\frac{\text{third term}}{\text{second term}}=\frac{d}{r}\ll 1$$

So we need the observation point to be a long way from the aperture.

If $d/\lambda>1$, the third term may become important. This is the regime of Fresnel diffraction. Fresnel diffraction occurs for

$$d\frac{d}{\lambda}\gtrsim r\gg d$$

Typically we find that Fresnel diffraction patterns are more complex than Fraunhoffer patterns. Most of the problems in the text refer to the Fraunhoffer regime.

3 Vectorial diffraction theory

A more careful analysis requires that we consider the vector nature of the fields. Thus we replace equation (2) with the vector relation:

$$\vec{E}\left(\vec{x}\right)=\int_S\left[\vec{E}\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)G-G\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)\vec{E}\right]da'$$

First we rewrite the integrand as:

$$2\vec{E}\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)G-\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)\left(G\vec{E}\right)$$

And then we convert the surface integral of the second term to a volume integral. Recall that the normal $\hat{\mathbf{n}}'$ is inward:

$$\begin{aligned}\int_s -\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)\left(G\tilde{\mathbf{E}}\right)da' &= \int_V \nabla^2\left(G\tilde{\mathbf{E}}\right)dV' \\ &= \int_V\left[\vec{\nabla}\left(\vec{\nabla}\cdot G\vec{E}\right)-\vec{\nabla}\times\left(\vec{\nabla}\times G\vec{E}\right)\right]dV \\ &= -\int_S\left[\hat{\mathbf{n}}'\left(\vec{\nabla}\cdot G\vec{E}\right)-\hat{\mathbf{n}}'\times\left(\vec{\nabla}\times G\vec{E}\right)\right]da'\end{aligned}$$

where

$$\vec{\nabla}\cdot G\vec{E} = G\vec{\nabla}\cdot\vec{E}+\vec{E}\cdot\vec{\nabla}G = 0+\vec{E}\cdot\vec{\nabla}G$$

since there is no charge density on S,and

$$\begin{aligned}\vec{\nabla}\times G\vec{E} &= \vec{\nabla}G\times\vec{E}+G\vec{\nabla}\times\vec{E} \\ &= \vec{\nabla}G\times\vec{E}+G\left(ik\vec{B}\right)\end{aligned}$$

Thus we have:

$$\begin{aligned}\vec{E}(\vec{x}) &= \int_S\left[2\vec{E}\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)G-\hat{n}'\left(\vec{E}\cdot\vec{\nabla}'G\right)+\hat{n}'\times\left(\vec{\nabla}'G\times\vec{E}+G\left(ik\vec{B}\right)\right)\right]da' \\ &= \int_S\left[2\vec{E}\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)G-\hat{n}'\left(\vec{E}\cdot\vec{\nabla}'G\right)+\vec{\nabla}'G\left(\hat{\mathbf{n}}'\cdot\vec{E}\right)-\vec{E}\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)G+Gik\hat{n}'\times\vec{B}\right]da' \\ &= \int_S\left[\vec{E}\left(\hat{\mathbf{n}}'\cdot\vec{\nabla}'\right)G-\hat{n}'\left(\vec{E}\cdot\vec{\nabla}'G\right)+\vec{\nabla}'G\left(\hat{\mathbf{n}}'\cdot\vec{E}\right)+ik\hat{n}'\times\vec{B}G\right]da' \\ &= \int_S\left[\left(\hat{\mathbf{n}}'\times\vec{E}\right)\times\vec{\nabla}'G+\left(\hat{\mathbf{n}}'\cdot\vec{E}\right)\vec{\nabla}'G+ik\hat{\mathbf{n}}'\times\vec{B}G\right]da'\end{aligned}$$

Now we can take $G = e^{ikR}/R$ and proceed as with the scalar theory. We can make the same Kirchhoff approximations, and the same inconsistencies remain. For r' very large, we have

$$G = \frac{e^{ikr'}}{r'}e^{-ik\hat{\mathbf{r}}'\cdot\vec{\mathbf{x}}} \simeq \frac{e^{ikr'}}{r'}$$

Now if we choose S_2 to be a large hemisphere of radius $r'\to\infty$, then $\hat{\mathbf{r}}' = -\hat{\mathbf{n}}'$, and

$$\vec{\nabla}'G = -ik\hat{\mathbf{n}}'G$$

and the integral over S_2 becomes:

$$\begin{aligned}&-ik\int_{S_2}\left[\left(\hat{\mathbf{n}}'\times\vec{E}\right)\times\hat{\mathbf{n}}'G+\left(\hat{\mathbf{n}}'\cdot\vec{E}\right)\hat{\mathbf{n}}'G-\hat{\mathbf{n}}'\times\vec{B}G\right]da' \\ = \;& -ik\int_{S_2}\left[\vec{E}-\hat{\mathbf{n}}'\times\vec{B}\right]Gda' = 0\end{aligned}$$

since

$$i\vec{k}\times\vec{E} = ik\vec{B}$$

for outgoing waves, and thus:

$$\hat{\mathbf{r}}'\times\vec{E} = \vec{B} = -\hat{\mathbf{n}}'\times\vec{E}$$

and so

$$\hat{\mathbf{n}}' \times \vec{B} = -\hat{\mathbf{n}}' \times \left(\hat{\mathbf{n}}' \times \vec{E}\right) = \vec{E}$$

on S_2. Thus the integral reduces to an integral over S_1.

Now if our observation point P is very distant from the sources on S_1,then as in the scalar theory we find $G = \frac{e^{ikR}}{4\pi R} \simeq \frac{e^{ikr}}{4\pi r} \exp\left(-i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{x}}'\right)$ and we expect the diffracted fields to have a similar form

$$\vec{E}_d = \frac{e^{ikr}}{r} \vec{F}\left(\vec{k}, \vec{k}_0\right)$$

Then the amplitude of the diffracted field may be written:

$$\begin{aligned} \vec{F}\left(\vec{k}, \vec{k}_0\right) &= \frac{i}{4\pi} \int_S \left[\left(\hat{\mathbf{n}}' \times \vec{E}\right) \times \left(-\vec{k}\right) + \left(\hat{\mathbf{n}}' \cdot \vec{E}\right)\left(-\vec{k}\right) + k\hat{\mathbf{n}}' \times \vec{B}\right] da' \\ &= \frac{i}{4\pi} \int_S \left[\vec{k} \times \left(\hat{\mathbf{n}}' \times \vec{E}\right) - \vec{k}\left(\hat{\mathbf{n}}' \cdot \vec{E}\right) + k\hat{\mathbf{n}}' \times \vec{B}\right] da' \end{aligned}$$

where the fields in the integral are also the diffracted fields. We know that $\vec{F}$ must be perpendicular to $\vec{k}$. In fact we can write the last two terms in the integrand, like the first, in the form $\vec{k} \times \vec{v}$ for some vector $\vec{v}$. First note that

$$\vec{k} \times \left[\vec{k} \times \left(\hat{\mathbf{n}}' \times \vec{B}\right)\right] = \vec{k}\left[\vec{k} \cdot \left(\hat{\mathbf{n}}' \times \vec{B}\right)\right] - k^2\left(\hat{\mathbf{n}}' \times \vec{B}\right)$$

and from Ampere's law:

$$\vec{k} \cdot \left(\hat{\mathbf{n}}' \times \vec{B}\right) = -\hat{\mathbf{n}}' \cdot \left(\vec{k} \times \vec{B}\right) = \hat{\mathbf{n}}' \cdot k\vec{E}$$

giving:

$$\begin{aligned} k\left(\hat{\mathbf{n}}' \times \vec{B}\right) &= \hat{\mathbf{k}}\left[\vec{k} \cdot \left(\hat{\mathbf{n}}' \times \vec{B}\right)\right] - \hat{\mathbf{k}} \times \left[\vec{k} \times \left(\hat{\mathbf{n}}' \times \vec{B}\right)\right] \\ &= \vec{k}\left(\hat{\mathbf{n}}' \cdot \vec{E}\right) - \hat{\mathbf{k}} \times \left[\vec{k} \times \left(\hat{\mathbf{n}}' \times \vec{B}\right)\right] \end{aligned}$$

so the terms in the integrand are:

$$k\hat{\mathbf{n}}' \times \vec{B} - \vec{k}\left(\hat{\mathbf{n}}' \cdot \tilde{\mathbf{E}}\right) = -\frac{1}{k}\vec{k} \times \left[\vec{k} \times \left(\hat{\mathbf{n}}' \times \vec{B}\right)\right]$$

and thus

$$\vec{F}\left(\vec{k}, \vec{k}_0\right) = \frac{-i}{4\pi k}\vec{k} \times \int_{S_1} \left[\vec{k} \times \left(\hat{\mathbf{n}}' \times \vec{B}\right) - \hat{\mathbf{n}}' \times \vec{E}\right] da'$$

The integrand also depends on $\vec{k}_0$ if we use the incident fields in the apertures.

4 Special case of plane conducting screen

Let the conducting screen be the $z = 0$ plane, with the incident fields propagating toward the screen from negative z. We can decompose the total electric field into three parts: the incident field $\vec{E}_0$, the field due to reflection by a solid conducting sheet, $\vec{E}_r$, and the diffracted field $\vec{E}_d$. Then

$$\vec{E} = \vec{E}_0 + \vec{E}_r + \vec{E}_d = \vec{E}_0 + \vec{E}_1$$

Then in Region II ($z > 0$) $\vec{E}_0 + \vec{E}_r = 0$.

The fields $\vec{E}_r + \vec{E}_d = \vec{E}_1$ are produced by currents in the conducting sheet:

$$\vec{j} = j_x \hat{\mathbf{x}} + j_y \hat{\mathbf{y}}$$

The resulting vector potential is given by:

$$\left(\nabla^2 + k^2\right) \vec{A} = \frac{4\pi}{c} \vec{j}$$

and consequently $A_z = 0$. Also, since the operator on the LHS is even in z, and the right hand side is zero except for $z = 0$, then $\vec{A}$ is even in z. Then

$$B_y = \left(\vec{\nabla} \times \vec{A}\right)_y = \frac{\partial A_x}{\partial z}$$

is odd in z. Similarly,

$$E_x = -\frac{\partial A_x}{\partial t}$$

is even in z. Thus we have

$$E_x \, , \; E_y, \text{ and } B_z \text{ are even in } z$$

and

$$E_z \, , \; B_x, \text{ and } B_y \text{ are odd in } z$$

The fields that are odd in z need not be zero at $z = 0$ where the conducting screen exists, because of the surface charge density and currents in the screen. But in the apertures, these odd fields must be zero. Thus in the apertures, $\vec{B}_{\text{tangential}}$ and E_{normal} are the incident fields alone.

Now the mathematical problem that determines $\vec{A}$ is a Neumann problem, because $\frac{\partial \vec{A}}{\partial z} = \vec{B}_{\text{tangential}}$ is determined (through Ampere's law) by the currents in the screen. Thus we must use the Neumann Green's function:

$$\vec{A} = -\frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \frac{\partial \vec{A}}{\partial z'} da'$$

$$\begin{aligned} A_x &= -\frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \frac{\partial A_x}{\partial z'} da' = -\frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} B_y \, da' \\ &= \frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{B}_1\right)_x da' \end{aligned}$$

A similar expression holds for A_y, and thus

$$\vec{B}_1 = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{B}_1\right) da' \tag{7}$$

Since $\hat{\mathbf{z}} \times \vec{B}_1 = 0$ in the aperture, the integral is over the *conducting material only*. We can evaluate the integral on either side of the screen. For $z < 0$, the normal becomes $-\hat{\mathbf{z}}$ instead of $\hat{\mathbf{z}}$, thus confirming our expectations about the oddness of $\vec{B}_{\text{tan}}$.

Now the source-free Maxwell equations are invariant under the transformation $\vec{B} \to \vec{E}$,

$\vec{E} \to -\vec{B}$, so we can write a similar expression for $\vec{E}_1$:

$$\vec{E}_1 = \vec{\nabla} \times \frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{E}_1 \right) da'$$

Now if we want the electric field for $z < 0$, we must change the sign in front of the integral as well as the sign in front of $\hat{\mathbf{z}}$ in order to obtain the correct symmetry. Since E_{1z} is zero in the aperture, we cannot simplify this integral as it stands. However:

$$\begin{aligned} \vec{E}_1 &= \vec{\nabla} \times \frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \left(\vec{E}_1 + \vec{E}_0 - \vec{E}_0 \right) \right) da' \\ &= \vec{\nabla} \times \frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{E} \right) da' - \vec{\nabla} \times \frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{E}_0 \right) da' \end{aligned}$$

where $\tilde{\mathbf{E}}$ is the total electric field, whose tangential component vanishes on the conducting surface. Thus the first integral is an integral over the *holes alone*. The second term

$$\vec{\nabla} \times \frac{1}{2\pi} \int_{screen} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{E}_0 \right) da' = \vec{E}_0 \text{ for } z > 0$$

Thus

$$\vec{E} = \vec{E}_1 + \vec{E}_0 = \vec{E}_d = \vec{\nabla} \times \frac{1}{2\pi} \int_{\text{holes}} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{E} \right) da' \tag{8}$$

where the field in the integrand is the total electric field in the aperture. In practical applications we can approximate by using the incident fields as we have indicated previously. Equation (8) is the vector Smythe-Kirchoff relation.

5 Babinet's principle

Babinet's principle relates the diffraction patterns due to 2 complementary screens S and S_c. The complementary screen S_c has apertures where S is solid and vice-versa. Thus the sum $S + S_c$ is a solid surface.

5.1 Scalar principle

For any complete closed surface we can write a mathematical identity for any scalar function ψ : equation (1) or, using the outgoing wave Green's function, equation (2) Thus:

$$\begin{aligned} \psi(\vec{x}) &= -\frac{1}{4\pi} \int_{S+S_c+S_2} \frac{e^{ikR}}{R} \left[\psi(\vec{x}') \, ik \left(1 + \frac{i}{kR} \right) \hat{\mathbf{n}} \cdot \hat{\mathbf{R}} + \hat{\mathbf{n}} \cdot \vec{\nabla}' \psi(\vec{x}') \right] da' \\ &= -\frac{1}{4\pi} \int_{S+S_c} \frac{e^{ikR}}{R} \left[\psi(\vec{x}') \, ik \left(1 + \frac{i}{kR} \right) \hat{\mathbf{n}} \cdot \hat{\mathbf{R}} + \hat{\mathbf{n}} \cdot \vec{\nabla}' \psi(\vec{x}') \right] da' \end{aligned}$$

where as usual, the integral over S_2 vanishes. This result is exact. Now, using the Kirchhoff approximation, the integral over S_c (the apertures in S) gives the diffracted field ψ_a in region II due to the screen S. Similarly the integral over S gives the diffracted field ψ_b due

to the complementary screen S_c. Thus

$$\psi = \psi_a + \psi_b$$

Now in directions where $\psi = 0$,we have $\psi_a = -\psi_b$, and thus the diffraction pattern, which depends on ψ^2, is the same. Thus for example the diffraction pattern due to a hole of radius a and due to a disk of radius a are the same except "straight ahead", where the incident field is non-zero.

5.2 Vector principle

A more careful statement can be made for a plane conducting screen. First we define the two situations that are complementary:

1. The incident fields are $\vec{E}_0$, $\vec{B}_0$ and the screen is S_0.
2. The incident fields are $\vec{E}_c = \vec{B}_0$, $\vec{B}_c = -\vec{E}_0$, and the complementary screen S_c.

Then we can use equation (8) to find the diffracted field in problem 1:

$$\vec{E}_1 = \vec{\nabla} \times \frac{1}{2\pi} \int_{\text{holes in S}_0} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{E}_1 \right) da'$$

For problem 2, we use equation (7):

$$\vec{B}_2 = \vec{\nabla} \times \frac{1}{2\pi} \int_{\text{conducting part of S}_c} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{B}_2 \right) da'$$

These two integrals are taken over the same surface. Since these two expressions are mathematically identical, then

$$\vec{E}_1 = \vec{B}_2$$

in region II. However the expression for $\vec{B}_2$ gives the field due to the currents in the sheet. The total magnetic field in region 2 is

$$\vec{B}_{2,\text{tot}} = \vec{B}_c + \vec{B}_2 = -\vec{E}_0 + \vec{E}_1.$$

The intensity of the radiation is given by $\left|\vec{E}\right|^2$ or equivalently $\left|\vec{B}\right|^2$. Thus the diffraction pattern is the same for both screens wherever $\vec{E}_0$ is zero.

A similar analysis shows that

$$\vec{E}_2 = \vec{\nabla} \times \frac{1}{2\pi} \int_{\text{holes in S}_c} \frac{e^{ikR}}{R} \left(\hat{\mathbf{z}} \times \vec{E}_2 \right) da' = -\vec{B}_1$$

where here again $\vec{B}_1$ is the field due to the sheet alone, and the minus sign appears so that both waves are outgoing. Thus

$$\begin{aligned} \vec{B}_{1,\text{tot}} &= \vec{B}_0 + \vec{B}_1 = \vec{B}_0 - \vec{E}_2 \\ &= \hat{k}_0 \times \vec{E}_0 - \vec{E}_2 \end{aligned}$$

which leads to the same prediction with regard to the diffraction pattern.

Babinet's principle is used in the design of microwave antennae. A slot in a conducting plate radiates the same pattern as a metal strip. Thus slots in wave-guides can serve as

effective microwave radiatiors.