Jackson notes 2020

Dielectrics

When an electric field is applied to a non-conducting medium, the charge distributions in the individual molecules are distorted, giving rise to a dipole moment $\vec{p}_i$ associated with each molecule of type i. Some materials have molecular dipoles even when no field is applied. In these materials the electric field aligns the dipoles parallel to $\vec{E}$ ("sphermult" notes page 12.) Then the electric polarization (dipole moment per unit volume) is

$$\vec{P}(\vec{x}) = \sum_i n_i(\vec{x})\,\vec{p}_i$$

where n_i is the number density of molecules of type i. The electric potential contributed by these dipoles is (sphermult notes eqn 9)

$$\Phi(\vec{x})_{\text{dipoles}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3x'$$

We can perform some of the usual tricks on the integral.

$$\begin{aligned}
\Phi(\vec{x})_{\text{dipoles}} &= \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \vec{P}\cdot\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|} d^3x' \\
&= \frac{1}{4\pi\varepsilon_0}\int \left\{\vec{\nabla}'\cdot\left(\frac{\vec{P}}{|\vec{x}-\vec{x}'|}\right) - \frac{\vec{\nabla}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|}\right\} d^3x' \\
&= \frac{1}{4\pi\varepsilon_0}\left\{\int_{S_\infty} \frac{\vec{P}}{|\vec{x}-\vec{x}'|}\cdot\hat{n}'\,dA' - \int \frac{\vec{\nabla}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|} d^3x'\right\} \\
&= \frac{1}{4\pi\varepsilon_0}\left\{-\int \frac{\vec{\nabla}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|} d^3x'\right\}
\end{aligned}$$

since $\vec{P} = 0$ on the surface at infinity. Compare this result with eqn (29) in Notes 1. The divergence of $\vec{P}$ acts as a "bound" charge density

$$\rho_B = -\vec{\nabla}\cdot\vec{P} \tag{1}$$

in producing potential.

The total potential due to (free) charge density ρ plus bound dipoles may be written:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\rho(\vec{x}') - \vec{\nabla}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|} d^3x'$$

We may write this result in differential form as

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\left(\rho(\vec{x}) - \vec{\nabla}\cdot\vec{P}\right) \tag{2}$$

If the medium is uniform, in most cases $\vec{\nabla} \cdot \vec{P} = 0$ except at boundaries.

Defining the electric displacement

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} \tag{3}$$

equation (2) becomes

$$\vec{\nabla} \cdot \vec{D} = \rho\left(\vec{x}\right) \tag{4}$$

Since this equation contains only the free charge density ρ, we do not have to concern ourselves with the distribution of $\vec{P}$.

If the response of the medium is linear and isotropic, then

$$\vec{P} = \varepsilon_0 \chi \vec{E} \tag{5}$$

where χ is the electric susceptibility of the medium. Then

$$\vec{D} = \varepsilon_0 \vec{E}\left(1 + \chi\right) = \varepsilon \vec{E} \tag{6}$$

where $\varepsilon/\varepsilon_0 = 1 + \chi$ is the dielectric constant of the medium.

If the medium is not isotropic, (χ depends on the direction of $\vec{E}$,) this equation (6) is replaced by the tensor relationship

$$D_i = \varepsilon_{ij} E_j$$

(See Jackson problem 7.16 for example. Also see
http://www.physics.sfsu.edu/~lea/courses/grad/plasmawavesi.PDF section 2.)

Boundary value problems with dielectrics

1. Method of images

A point charge Q is placed in a uniform medium of dielectric constant $\kappa_1 = \varepsilon_1/\varepsilon_0$, at a distance d from a plane boundary with a medium of dielectric constant $\kappa_2 = \varepsilon_2/\varepsilon_0$. We want to find the electric field everywhere.

First note that we may place "image" charges outside medium 1 and still satisfy the differential equation

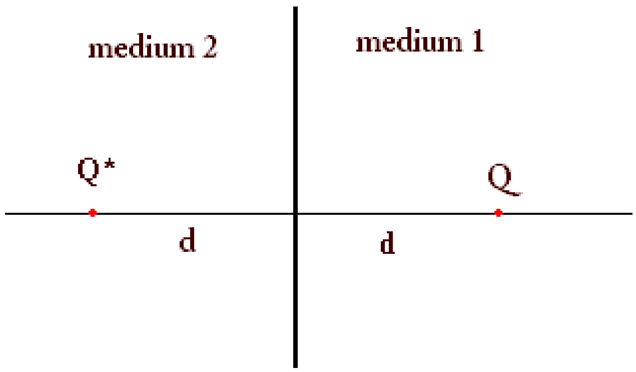
$$\vec{\nabla} \cdot \vec{D}_1 = \varepsilon_1 \vec{\nabla} \cdot \vec{E}_1 = -\varepsilon_1 \nabla^2 \Phi_1 = -Q\delta\left(\vec{x} - \vec{x}_1\right)$$

in medium 1. Similarly, we may put "image" charges outside medium 2 and still satisfy the differential equation

$$\vec{\nabla} \cdot \vec{D}_2 = \varepsilon_2 \vec{\nabla} \cdot \vec{E}_2 = -\varepsilon_2 \nabla^2 \Phi_2 = 0$$

in medium 2. We can use these images to satisfy the boundary conditions, just as we did with conductors (notes 3 §1.1).

We place the $z-$axis along the line joining the two charges, with origin on the boundary. The system has rotational symmetry about this line, so all image charges must lie on the $z-$axis. Using our experience with conductors as a guide, we guess that the image to the left of the boundary is a charge Q^* the same distance d from the boundary as Q. The potential in medium 1 is computed assuming that medium 1 extends to the left of the boundary.



We use cylindrical coordinates (ρ, ϕ, z), so

$$\Phi_1(z > 0) = \frac{1}{4\pi\varepsilon_1}\left(\frac{Q}{R_1} + \frac{Q^*}{R_2}\right)$$

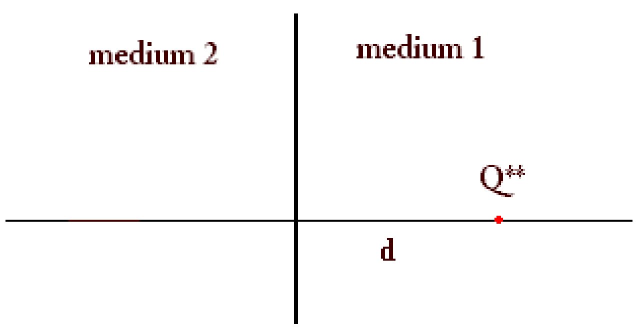
where

$$R_1^2 = (z-d)^2 + \rho^2$$

and

$$R_2^2 = (z+d)^2 + \rho^2$$

Similarly, when computing the fields in medium 2, we use a charge Q^{**}, located to the right of the boundary and at distance d from it:



Then the potential in medium 2 is computed as if medium 2 extends to the right of the boundary:

$$\Phi_2\,(z < 0) = \frac{1}{4\pi\varepsilon_2}\frac{Q^{**}}{R_1}$$

Now we calculate the fields and apply the boundary conditions. For $z > 0$,

$$\begin{aligned}
\vec{E}_1 &= -\vec{\nabla}\Phi_1 \\
&= \frac{1}{4\pi\varepsilon_1}\left[\frac{Q}{R_1^3}\left(\rho\ \hat{\rho} + (z-d)\ \hat{z}\right) + \frac{Q^*}{R_2^3}\left(\rho\ \hat{\rho} + (z+d)\ \hat{z}\right)\right]
\end{aligned}$$

and for $z < 0$

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_2}\frac{Q^{**}}{R_1^3}\left(\rho\ \hat{\rho} + (z-d)\ \hat{z}\right)$$

At the boundary $z = 0$, $R_1 = R_2$, and normal $\vec{D}$ and tangential $\vec{E}$ are continuous (Notes 1 eqns 10 and 15 with zero free charge density at the boundary). The normal component of $\vec{D}$ is D_z, so

$$Q - Q^* = Q^{**}$$

and the tangential component of $\vec{E}$ is E_ρ, so

$$\frac{Q + Q^*}{\varepsilon_1} = \frac{Q^{**}}{\varepsilon_2}$$

Thus

$$Q - Q^* = \frac{\varepsilon_2}{\varepsilon_1}\left(Q + Q^*\right)$$

so

$$Q^* = Q\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

and

$$Q^{**} = Q\left(1 - \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) = Q\frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

Note that $Q^* \to -Q$ as $\varepsilon_2 \to \infty$, the conductor limit. Also $Q^* \to 0$ and $Q^{**} \to Q$ as $\varepsilon_2 \to \varepsilon_1$, as expected.

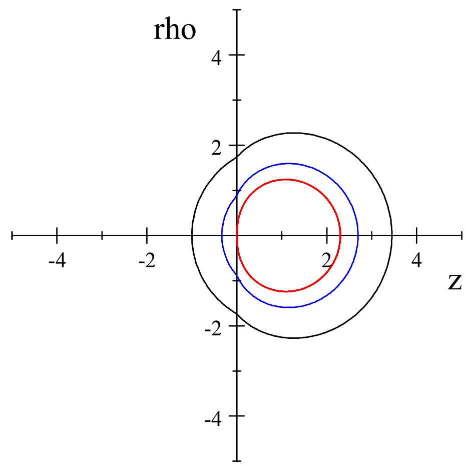
With these values, the fields are:

$$\begin{aligned}
\vec{E}_1 &= \frac{Q}{4\pi\varepsilon_1}\left[\frac{\rho\hat{\rho} + (z-d)\,\hat{z}}{R_1^3} + \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)\frac{\rho\hat{\rho} + (z+d)\,\hat{z}}{R_2^3}\right] \\
&= \frac{Q}{4\pi\varepsilon_1}\left\{\vec{\rho}\left[\frac{1}{R_1^3} + \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + \varepsilon_2)}\frac{1}{R_2^3}\right] + \left[\frac{z-d}{R_1^3} + \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + \varepsilon_2)}\frac{(z+d)}{R_2^3}\right]\hat{z}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\vec{E}_2 &= \frac{Q}{4\pi\varepsilon_2}\frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\left(\frac{\rho\hat{\rho} + (z-d)\,\hat{z}}{R_1^3}\right) \\
&= \frac{Q}{2\pi}\frac{1}{\varepsilon_1 + \varepsilon_2}\left(\frac{\rho\hat{\rho} + (z-d)\,\hat{z}}{R_1^3}\right)
\end{aligned}$$

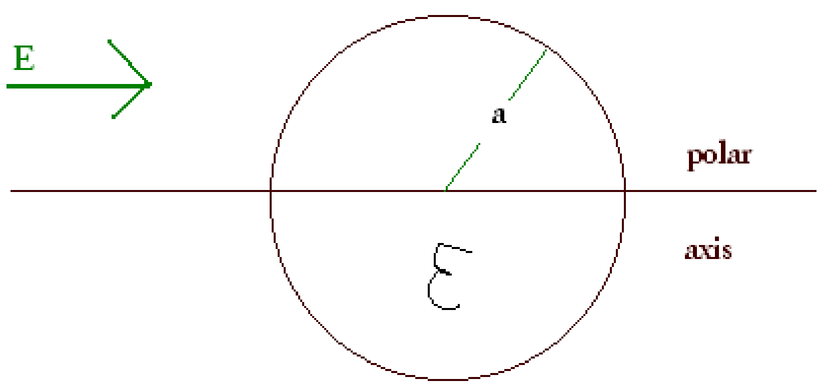
Thus the field in medium 2 looks like that due to a charge Q in a uniform medium with the average dielectric constant. Both results are correct in the limit $\varepsilon_2 = \varepsilon_1$.



Plot of the equipotentials for $\varepsilon_2 = 2\varepsilon_1$, All distances are scaled by d. The scaled potential is $U = 4\pi\varepsilon_1\Phi d/Q$. $U = 1/3$ (black line) $U = 1/2$ (blue line) $U = 2/3$ (red line)

2. Dielectric sphere in a uniform field

We use spherical coordinates with origin at the center of the sphere, and put the polar axis along the direction of the uniform field $\vec{E}_0$, which is the only direction singled out in the system.



The system is rotationally symmetric about this axis, and the only place where there is any net charge density (free or bound) is on the surface of the sphere.

There is also charge at infinity to produce the uniform field $\vec{E}_{0.}$ Thus, inside the sphere, the potential satisfies Laplace's equation and has the form

$$\Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\mu)$$

The terms in r^{-l-1} have been excluded because we expect the potential to be finite at the origin. Outside the sphere

$$\Phi_{\text{out}} = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l P_l(\mu)}{r^{l+1}}$$

The first term in Φ_{out} describes the uniform field at infinity[1]. All other terms in the potential (which are due to the bound surface charge density on the sphere) should go to zero as $r \to \infty$, so we excluded additional positive powers of r. Applying the boundary conditions at $r = a$:

Normal (radial) component of $\vec{D}$ is continuous:

$$D_r = -\varepsilon \frac{\partial \Phi}{\partial r}$$

$$-\frac{\varepsilon}{\varepsilon_0} \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\mu) = E_0 \cos\theta + \sum_{l=0}^{\infty} (l+1) \frac{B_l P_l(\mu)}{a^{l+2}} \tag{7}$$

Tangential $\vec{E}$ (or, equivalently, potential) is continuous.

$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}$$

We may use the μ derivative rather than θ.

$$\sum_{l=0}^{\infty} A_l a^{l-1} P_l'(\mu) = -E_0 + \sum_{l=0}^{\infty} \frac{B_l P_l'(\mu)}{a^{l+2}} \tag{8}$$

Using orthogonality of the Legendre polynomials P_l, we may equate each term in the sum in equation (7)

$$\begin{aligned}
\frac{\varepsilon}{\varepsilon_0} l A_l a^{l-1} &= -(l+1)\frac{B_l}{a^{l+2}} & l > 1 \\
\frac{\varepsilon}{\varepsilon_0} A_1 &= -E_0 - 2\frac{B_1}{a^3} & l = 1 \\
0 &= B_0 & l = 0
\end{aligned}$$

Interestingly, the P_l' are also orthogonal (see Lea Ch 8 probs 4 and 5). Thus we may also equate each term in the sum in equation (8):

$$\begin{aligned}
A_l a^{l-1} P_l'(\mu) &= \frac{B_l P_l'(\mu)}{a^{l+2}} & l > 1 \\
A_1 &= -E_0 + \frac{B_1}{a^3} & l = 1
\end{aligned}$$

[1] We did something very similar when computing current flow. See currentflow notes pg 2.

(Note: we could get the same result from continuity of the potential.) For $l > 1$, the only possible solution[2] is $A_l = B_l \equiv 0$, while for $l = 1$:

$$\begin{aligned} -E_0 - 2\frac{B_1}{a^3} &= \frac{\varepsilon}{\varepsilon_0}\left(-E_0 + \frac{B_1}{a^3}\right) \\ B_1 &= E_0 a^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \end{aligned}$$

and then

$$A_1 = -E_0 + E_0 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} = -3\frac{\varepsilon_0 E_0}{\varepsilon + 2\varepsilon_0}$$

We have not found A_0. Continuity of the potential requires that it be zero. Thus

$$\Phi_{\text{in}} = -\frac{3\varepsilon_0 E_0}{\varepsilon + 2\varepsilon_0} r\cos\theta = -\frac{3\varepsilon_0 E_0}{\varepsilon + 2\varepsilon_0} z$$

and

$$\Phi_{\text{out}} = -E_0 r\cos\theta + E_0 a^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\frac{\cos\theta}{r^2}$$

The field inside the sphere is uniform:

$$\vec{E}_{\text{in}} = E_0 \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}\hat{z}$$

and outside it is a superposition of the uniform field at infinity and a dipole field ("sphermult" notes eqn 9). The dipole moment is found from

$$\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = E_0 a^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\frac{\cos\theta}{r^2}$$

so that

$$\vec{p} = 4\pi\varepsilon_0 \vec{E}_0 a^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} = \frac{4\pi}{3}a^3\vec{P}$$

where the polarization is

$$\vec{P} = \frac{3\varepsilon_0(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0}\vec{E}_0 = (\varepsilon - \varepsilon_0)\vec{E}_{\text{in}} = \varepsilon_0\chi\vec{E}_{\text{in}}$$

as expected. Since $\vec{\nabla}\cdot\vec{P} = 0$ inside the sphere, all the bound charge density is on the surface, as predicted.

Notice that these results have the right limits in the cases $\varepsilon \to \varepsilon_0$ (no sphere) and $\varepsilon \to \infty$ (conducting sphere). If $\varepsilon > \varepsilon_0$ (the case for most ordinary materials), $E_{\text{in}} < E_0$. The results are important in computing scattering of radiation by small spheres.

[2] We have seen this result before. See current flow notes page 3.

Energy

For a linear medium, the electric energy is given by (Notes 2 eqn 5)

$$U = \frac{1}{2}\int \vec{E}\cdot\vec{D}\;dV \tag{9}$$

It is interesting to consider what happens when we *change* the properties of the medium, for example, if we introduce a dielectric where intially there was vacuum.

First let us consider the case of fixed sources. That is, the free charge density function $\rho_0(\vec{x})$ remains unchanged. The initial energy is

$$U_0 = \frac{1}{2}\int \vec{E}_0\cdot\vec{D}_0\;dV$$

and after introducing the dielectric, the fields change and

$$U_f = \frac{1}{2}\int \vec{E}\cdot\vec{D}\;dV$$

The change in energy is thus

$$\Delta U = \frac{1}{2}\int\left(\vec{E}\cdot\vec{D}\;-\vec{E}_0\cdot\vec{D}_0\right)\;dV \tag{10}$$

It seems obvious that the difference in energy is associated with the polarization of the medium, so let's see how. First we rewrite the integrand as follows:

$$\vec{E}\cdot\vec{D}\;-\vec{E}_0\cdot\vec{D}_0 = \left(\vec{E}+\vec{E}_0\right)\cdot\left(\vec{D}-\vec{D}_0\right)+\vec{E}\cdot\vec{D}_0\;-\vec{E}_0\cdot\vec{D} \tag{11}$$

Next we show that the integral of the first term is zero. The initial and final states are time independent, so $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\vec{E}_0 = 0$, and we may write $\vec{E}+\vec{E}_0 = -\vec{\nabla}\Psi$ for some scalar function Ψ. Then the integral of the first term in (11) is

$$\begin{aligned} I_1 &= \int_V\left(\vec{E}+\vec{E}_0\right)\cdot\left(\vec{D}-\vec{D}_0\right)\;dV = -\int_V\vec{\nabla}\Psi\cdot\left(\vec{D}-\vec{D}_0\right)\;dV \\ &= -\int_V\vec{\nabla}\cdot\left[\Psi\left(\vec{D}-\vec{D}_0\right)\right]\;dV - \int_V\Psi\vec{\nabla}\cdot\left(\vec{D}-\vec{D}_0\right)\;dV \\ &= -\int_{S_\infty}\Psi\left(\vec{D}-\vec{D}_0\right)\cdot\hat{n}\;dA - 0 \end{aligned}$$

where we used the divergence theorem in the first term and the fact that $\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot\vec{D}_0 = \rho_0$ in the second term. Now we argue that the surface integral is zero for all the usual reasons.

We have now reduced the change in energy (10) to

$$\begin{aligned}
\Delta U &= \frac{1}{2}\int \left(\vec{E}\cdot\vec{D}_0 \ -\vec{E}_0\cdot\vec{D}\right)\ dV \\
&= \frac{1}{2}\int \left[\vec{E}\cdot\left(\varepsilon_0\vec{E}_0\right) \ -\vec{E}_0\cdot\left(\varepsilon_0\vec{E}+\vec{P}\right)\right]\ dV \qquad \text{(equation 3)} \\
&= -\frac{1}{2}\int_{V_1}\vec{P}\cdot\vec{E}_0\ dV \qquad (12)
\end{aligned}$$

where the integral is now over the volume V_1 containing the dielectric, where $\varepsilon\neq\varepsilon_0$ and $\vec{P}\neq 0$.

The result is not surprising since we have already found the energy of a dipole in an *external* field to be $-\vec{p}\cdot\vec{E}$ (multipole notes eqn 23). The factor 1/2 is the usual factor that avoids double-counting.

Equation (12) shows that the energy is reduced when $\vec{P}$ is parallel to $\vec{E}_0$ – the usual case for ordinary materials. The stored energy is reduced because the fields do work to create and/or align the dipoles. This means that a dielectric object will be drawn toward regions of higher field. (See eg LB example 27.9 with the battery disconnected.)

Now let's consider the forces acting. If a dielectric body undergoes a displacement $\delta\xi$ with a corresponding change in energy δU, and the sources of the fields (the charges) are kept fixed, then the system is isolated, and the work done *by* the system reduces its energy,

$$\delta U = -\vec{F}\cdot d\vec{\xi} = -F_\xi d\xi$$

Then the $\xi-$component of the force exerted on the body by the fields is

$$F_\xi = -\left.\frac{\partial U}{\partial\xi}\right|_Q \qquad (13)$$

where the derivative is taken at constant Q. This result is of a familar form.

Often we find that potentials are kept fixed (the battery is kept connected in LB Example 27.9), and the battery acts as a source or sink of charge, and also of *energy.* In this case the system is *not* isolated. So we have to compute the energy changes and forces resulting in two steps.

We go back to the fundamental relation (704notes_2 eqn 3)

$$U = \frac{1}{2}\int \rho\left(\vec{x}\right)\Phi\left(\vec{x}\right)\ dV, \qquad (14)$$

where ρ is the free charge density, to compute the change in energy.

Then the change in energy is

$$\delta U = \frac{1}{2}\int \left[\delta\rho\left(\vec{x}\right)\Phi\left(\vec{x}\right)+\rho\left(\vec{x}\right)\delta\Phi\left(\vec{x}\right)\right]\ dV$$

where the integral is over the conductors, because there is no free charge in the dielectric.

1. The conducting surfaces (labelled with n) are disconnected from the batteries, and then the total charge on each must remain fixed. Since each conductor is an equipotential, it does not matter if the charge *distribution* on each changes. We change the dielectric properties of the system to get an energy change δU_1 where

$$\delta U_1 = \sum_n \frac{1}{2}\delta\Phi_{1,n} q_n = \frac{1}{2}\int \rho(\vec{x})\,\delta\Phi_1(\vec{x})\; dV$$

2. The batteries are reconnected, allowing charge to flow on or off the conductors, with a consequent energy flow. The potentials return to their original values

$$\delta\Phi_2 = -\delta\Phi_1$$

and the energy change is

$$\delta U_2 = \frac{1}{2}\int \left[\delta\rho(\vec{x})\,\Phi(\vec{x}) + \rho(\vec{x})\,\delta\Phi_2(\vec{x})\right]\; dV$$

In this step we are not changing the dielectric properties of the system at all. Thus $\delta\Phi \propto \delta\rho$ (notes 1 eqn 29) and the two terms are equal, so we may also write

$$\begin{aligned}\delta U_2 &= \int \rho(\vec{x})\,\delta\Phi_2(\vec{x})\; dV \\ &= -\int \rho(\vec{x})\,\delta\Phi_1(\vec{x})\; dV = -2\delta U_1\end{aligned}$$

Thus the total energy change is

$$\delta U_1 + \delta U_2 = -\delta U_1,$$

the opposite of the previous value. The force exerted on the dielectric in a given state is the same no matter how the system came to be in that state, so in this case

$$F_\xi = +\left.\frac{\partial U}{\partial \xi}\right|_V \tag{15}$$

Compare with (13).

We can understand this result once again by thinking about a simple system. If a dielectric slab is inserted into a parallel plate capacitor, the force on the slab draws it in, whether or not the battery is connected. You can see this from a simple picture of the fields and charges. However, the energy stored increases if the battery is connected but decreases if it is not.