Example using Bessel functions– Sp 2020

# Circular wave guide

Let's investigate the propagation of waves in a wave guide that has a circular cross-section of radius $a$ filled with air. The $z-$axis runs along the cylinder axis. As usual we take

$$\vec{E} \propto e^{ikz}e^{-i\omega t}$$

to obtain the differential equation for the TM mode (waveguide notes eqn 21)

$$\left(\nabla_{\mathrm{t}}^2 + \gamma^2\right) E_z = 0$$

Then (waveguide notes eqn 20 and Table on pg 7)

$$\vec{E}_t = \frac{ik}{\gamma^2}\vec{\nabla}_{\mathrm{t}} E_z \tag{1}$$

with (waveguide notes eqn 22, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$)

$$\gamma^2 = \frac{\omega^2}{c^2} - k^2 \tag{2}$$

In polar coordinates the differential equation takes the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \gamma^2 E_z = 0$$

We look for a solution of the form

$$E_z = R(\rho)\,W(\phi)$$

and separate:

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{W}\frac{\partial^2 W}{\partial\phi^2} + \gamma^2\rho^2 = 0$$

The middle term is a function of $\phi$ only, while the other two terms are a function of $\rho$ only. Thus, setting the middle term equal to $-m^2$, we get

$$W = e^{im\phi}$$

and the remaining equation for $R$ is Bessel's equation of order $m$. (Lea eqn 8.69 or bessel notes eqn 1, with $k \to \gamma$)

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - m^2 R + \gamma^2\rho^2 R = 0$$

We need a solution that is finite at $\rho = 0$, so we choose $J$. Thus the solutions are

$$E_{z,m} = J_m(\gamma\rho)\,e^{im\phi}$$

The boundary condition is (waveguide notes eqn 10)

$$E_z = 0 \quad \text{for } \rho = a$$

so $\gamma a$ must be one of the roots $x_{mn}$ of $J_m(x)$. Then

$$E_z = \sum_{m,n} A_{mn} J_m \left( x_{mn} \frac{\rho}{a} \right) e^{im\phi} e^{ikz} e^{-i\omega t} \tag{3}$$

We would need more information about how the guide is excited to find the coefficients $A_{mn}$.

We find the other field components from (equation 1)

$$\begin{aligned}
\vec{E}_{\mathrm{t},mn} &= i\frac{k}{\gamma^2}\vec{\nabla}_t E_{z,mn} = i\frac{ka^2}{x_{mn}^2}\left(\frac{\partial E_{z,mn}}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial E_{z,mn}}{\partial \phi}\hat{\phi}\right) \\
&= i\frac{ka^2}{x_{mn}^2}A_{mn}\left[\frac{x_{mn}}{a}J_m'\left(x_{mn}\frac{\rho}{a}\right)e^{im\phi}\hat{\rho} + \frac{im}{\rho}J_m\left(x_{mn}\frac{\rho}{a}\right)e^{im\phi}\hat{\phi}\right]e^{ikz}e^{-i\omega t} \\
&= \frac{ka}{x_{mn}}A_{mn}\left[iJ_m'\left(x_{mn}\frac{\rho}{a}\right)e^{im\phi}\hat{\rho} - \frac{m}{x_{mn}}\frac{a}{\rho}J_m\left(x_{mn}\frac{\rho}{a}\right)e^{im\phi}\hat{\phi}\right]e^{ikz}e^{-i\omega t} \qquad (4)
\end{aligned}$$

Taking the real part, we have the physical field:

$$\begin{aligned}
\vec{E}_{mn} &= -\frac{kaA_{mn}}{x_{mn}}\left[J_m'\left(x_{mn}\frac{\rho}{a}\right)\sin(kz + m\phi - \omega t)\hat{\rho} + \frac{m}{x_{mn}}\frac{a}{\rho}J_m\left(x_{mn}\frac{\rho}{a}\right)\cos(kz + m\phi - \omega t)\hat{\phi}\right] \\
&\quad + A_{mn}J_m\left(x_{mn}\frac{\rho}{a}\right)\cos(kz + m\phi - \omega t)\hat{z}
\end{aligned}$$

The roots are: $x_{0n} = 2.4,\ 5.5,\ 8.6, \cdots$
$x_{1n} = 3.8,\ 7.0, \cdots$ etc (Jackson p114).

The lowest root is $x_{01} = 2.4$. Thus the cutoff frequency ($k = 0$) for the TM modes is (from eqn. 2 with $k = 0$)

$$\frac{\omega_{\mathrm{c,TM}}}{c} = \gamma_{\min} = \frac{x_{\min}}{a} = \frac{x_{01}}{a} = \frac{2.4}{a}$$

The lowest frequency ($m = 0$) mode has no $\phi$ dependence and $\vec{E}_{\mathrm{t}}$ is purely radial. Since $J_0' = -J_1$ and $J_1(0) = 0$, $\vec{E}_{\mathrm{t},0n} \to 0$ at the center of the guide, as it must. (Remember: field lines can't cross.)

$$\vec{E}_{01} = A_{01}\left\{J_0\left(2.4\frac{\rho}{a}\right)\hat{z}\cos(kz - \omega t) + \frac{ka}{2.4}J_1\left(2.4\frac{\rho}{a}\right)\hat{\rho}\sin(kz - \omega t)\right\}$$

with (eqn 2)

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{2.4^2}{a^2}}$$

The next highest cutoff is for $m = 1$, $n = 1$ with $\omega_{c,11} = 3.8c/a$. The fields in the $m = 1$, $n = 1$ mode are

$$\begin{aligned}
\vec{E}_{11} &= A_{11}\left\{\left[\hat{z} - \hat{\phi}\frac{ka}{3.8}\frac{a}{3.8\rho}\right]J_1\left(3.8\frac{\rho}{a}\right)\cos(\phi + kz - \omega t)\right. \\
&\qquad \left. - \frac{ka}{3.8}\frac{J_0\left(3.8\frac{\rho}{a}\right) - J_2\left(3.8\frac{\rho}{a}\right)}{2}\hat{\rho}\sin(\phi + kz - \omega t)\right\}
\end{aligned}$$

where we used Lea eqn 8.90 for $J_1'$, and with

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{3.8^2}{a^2}}$$

At the guide center, $\rho = 0$, (to evaluate the $\phi$ component, remember that $J_1(x) = \frac{x}{2} + \cdots$ and take the limit of $J_1(x)/x$ as $x \to 0$).

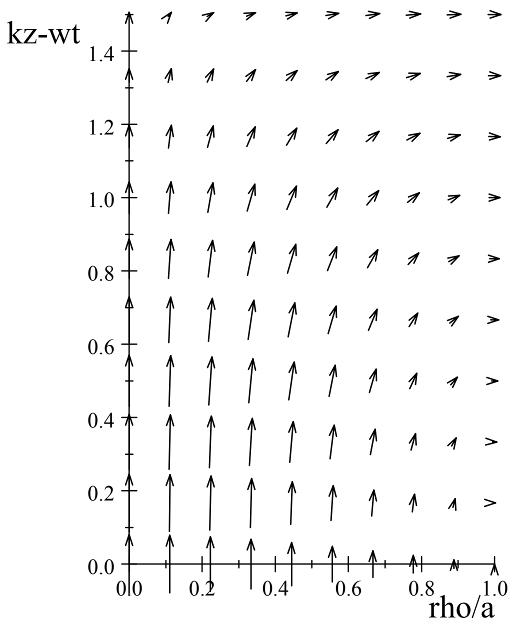
$$\begin{aligned}
\vec{E}_{11}(0) &= A_{11}\left\{-\frac{ka}{3.8}\left(\frac{1}{2}\hat{\rho}\sin(\phi + kz - \omega t) + \frac{1}{2}\hat{\phi}\cos(\phi + kz - \omega t)\right)\right\} \\
&= -\frac{ka}{7.6}A_{11}\left\{\hat{\rho}\left[\sin\phi\cos(kz - \omega t) + \cos\phi\sin(kz - \omega t)\right]\right. \\
&\qquad\qquad \left.+\hat{\phi}\left[\cos\phi\cos(kz - \omega t) - \sin\phi\sin(kz - \omega t)\right]\right\} \\
&= -\frac{ka}{7.6}A_{11}\left\{\cos(kz - \omega t)\left(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi\right) + \sin(kz - \omega t)\left(\hat{\rho}\cos\phi - \hat{\phi}\sin\phi\right)\right\} \\
&= -\frac{ka}{7.6}A_{11}\left\{\hat{y}\cos(kz - \omega t) + \hat{x}\sin(kz - \omega t)\right\}
\end{aligned}$$

The field makes an angle $kz - \omega t$ with the $y$ axis and, at a fixed $z$, rotates counter-clockwise in time.

If $\omega > 3.8c/a$, $\omega$ is also $> 2.4c/a$, and so the $m = 0$, $n = 1$ mode will exist as well. However, we are still below the cutoff for the $m = 0$, $n = 2$ mode. until $\omega > 5.5c/a$.

Convince yourself that we have satisfied all the boundary conditions and that the fields make sense. I will leave it to you to calculate $\vec{B}$. .

The diagram shows $\vec{E}$ in the $m = 0$, $n = 1$ mode. Note that the field lies along the axis at $\rho = 0$ and is perpendicular to the surface at $\rho = a$.


kz-wt
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
rho/a