Physics 704 Notes Sp 2020

1 Current Flow Problems

The current density $\vec{j}$ satisfies the charge conservation equation (notes 1 eqn 7)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \tag{1}$$

and thus in a steady state, $\vec{j}$ is solenoidal:

$$\vec{\nabla} \cdot \vec{j} = 0 \tag{2}$$

In a conducting medium, we may relate $\vec{j}$ to the electric field through the conductivity σ :

$$\vec{j} = \sigma \vec{E} = -\sigma \vec{\nabla} \Phi$$

At a boundary between two different media, integration of equation (2) over a pillbox straddling the boundary, as in notes _1 page 5, gives

$$\vec{j} \cdot \hat{n} \text{ is continuous across the boundary} \tag{3}$$

and continuity of tangential $\vec{E}$ also gives

$$\hat{t} \cdot \vec{\nabla} \Phi \text{ is continuous across the boundary}$$

which in most cases is equivalent to[1]

$$\Phi \text{ is continuous across the boundary} \tag{4}$$

Notice that the boundary condition on normal $\vec{j}$ (3) implies that E_{norm} is NOT continuous, and thus there must be charge on the boundary (Notes _1 eqn 11). This charge is the source of the fields that direct the current flow.

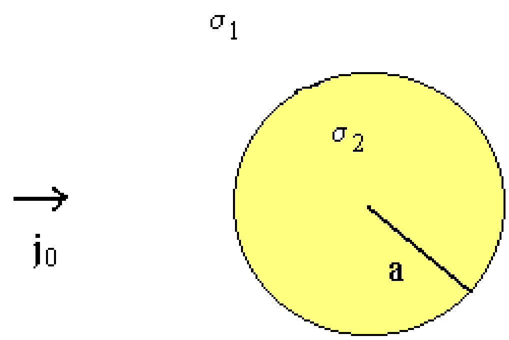
1.1 Example

An infinite, plane, conducting sheet with conductivity σ_1 contains a circular region of a different metal with conductivity σ_2 and radius a. Current enters the sheet at $x = -\infty$ flowing in the positive $x-$direction

$$\vec{j}\,(x \to -\infty) = j_0 \hat{x}$$

Find the pattern of current flow in the sheet.

[1] See Jackson §1.6 for a discussion of cases where Φ is not continuous.



First note that if $\sigma_1 < \sigma_2$, we expect current to flow inwards through the more conducting circular region, but if $\sigma_1 > \sigma_2$, we expect current to flow around the more resistive "obstacle" (current follows the path of least resistance).

In both of the regions $\rho > a$ and $\rho < a$ (but not *at* $\rho = a$ because of the charge there) the potential satisfies Laplace's equation, and thus the solution is of the form (Jackson eqn 2.69, page 76, notes3half pg 9). In the inner region we exclude the logarithmic term and the negative powers of ρ because they diverge at the origin.

$$\Phi_2(\rho < a) = \sum_{m=1}^{\infty} \rho^m (c_m \cos m\phi + b_m \sin m\phi)$$

In the outer region, we need a uniform electric field $\vec{E}_0 = \vec{j}_0/\sigma_1$ in the $x-$direction to drive the current at infinity. The potential $-j_0 x/\sigma_1 = -\frac{j_0}{\sigma_1}\rho\cos\phi$ describes this field. This corresponds to the $m = 1$ term in the potential. We exclude the logarithmic term and the other positive powers of ρ because they diverge at infinity. Thus

$$\Phi_1(\rho > a) = -\frac{j_0 \rho \cos\phi}{\sigma_1} + \sum_{m=1}^{\infty} \rho^{-m} (d_m \cos m\phi + e_m \sin m\phi)$$

The sum in this expression represents the potential due to the charge on the boundary at $\rho = a$.

Next we apply the boundary conditions at $\rho = a$.

Continuity of Φ :

$$\sum_{m=1}^{\infty} a^m (c_m \cos m\phi + b_m \sin m\phi) = -\frac{j_0 a \cos\phi}{\sigma_1} + \sum_{m=1}^{\infty} a^{-m} (d_m \cos m\phi + e_m \sin m\phi)$$

Making use of the orthogonality of the cosines and sines, we may equate term by term to get

$$\begin{aligned} a^m c_m &= a^{-m} d_m \qquad m > 1 && (5) \\ a c_1 &= -\frac{j_0 a}{\sigma_1} + \frac{d_1}{a} && (6) \end{aligned}$$

and

$$a^m b_m = a^{-m} e_m \tag{7}$$

Continuity of $j_n = j_\rho = \sigma E_\rho = -\sigma \partial\Phi/\partial\rho$:

$$-\sigma_2 \sum_{m=1}^{\infty} m a^{m-1} \left(c_m \cos m\phi + b_m \sin m\phi\right) = j_0 \cos\phi + \sigma_1 \sum_{m=1}^{\infty} m a^{-m-1} \left(d_m \cos m\phi + e_m \sin m\phi\right)$$

Using orthogonality of the trig functions, we have:

$$\begin{aligned} -\sigma_2 m a^{m-1} c_m &= \sigma_1 m a^{-m-1} d_m \qquad m > 1 && (8) \\ -\sigma_2 c_1 &= j_0 + \sigma_1 a^{-2} d_1 \qquad m = 1 && (9) \end{aligned}$$

and

$$-\sigma_2 m a^{m-1} b_m = \sigma_1 m a^{-m-1} e_m \tag{10}$$

The only solution to equations (5) and (8) is $c_m = d_m = 0,\ m > 1$. Similarly, from equations (7) and (10), $b_m = e_m = 0$. We should have expected this result because the input to the system (the current at infinity) is an $m = 1$ mode. The remaining equations (6) and (9) give

$$\begin{aligned} -\sigma_2 c_1 &= -\sigma_2 \left(-\frac{j_0}{\sigma_1} + \frac{d_1}{a^2}\right) = j_0 + \sigma_1 \frac{d_1}{a^2} \\ \sigma_2 \frac{j_0}{\sigma_1} - j_0 &= \sigma_2 \frac{d_1}{a^2} + \sigma_1 \frac{d_1}{a^2} \end{aligned}$$

So

$$d_1 = \frac{j_0 a^2}{\sigma_1} \frac{(\sigma_2 - \sigma_1)}{\sigma_2 + \sigma_1} \tag{11}$$

and then from (6),

$$\begin{aligned} c_1 &= -\frac{j_0}{\sigma_1} + \frac{d_1}{a^2} = -\frac{j_0}{\sigma_1} + \frac{j_0}{\sigma_1} \frac{(\sigma_2 - \sigma_1)}{\sigma_2 + \sigma_1} \\ &= -\frac{2 j_0}{\sigma_2 + \sigma_1} \end{aligned}$$

Thus the potential is

$$\Phi_2 (\rho < a) = -\frac{2 j_0}{\sigma_2 + \sigma_1} \rho \cos\phi = -\frac{2 j_0}{\sigma_2 + \sigma_1} x$$

and

$$\Phi_1 (\rho > a) = -\frac{j_0 \rho \cos\phi}{\sigma_1} + \frac{a^2}{\rho} \frac{j_0}{\sigma_1} \frac{(\sigma_2 - \sigma_1)}{\sigma_2 + \sigma_1} \cos\phi$$

The current is given by

$$
\begin{aligned}
\vec{j} &= -\sigma\vec{\nabla}\Phi = \frac{2\sigma_2}{\sigma_2+\sigma_1} j_0\hat{x} \quad \rho < a \\
&= j_0\hat{x} + j_0\frac{(\sigma_2-\sigma_1)}{\sigma_2+\sigma_1}\frac{a^2}{\rho^2}\left(\hat{\rho}\cos\phi + \hat{\phi}\sin\phi\right) \quad \rho > a
\end{aligned}
$$

The current for $\rho < a$ is uniform, and $\left|\vec{j}\right|$ is $> j_0$ if $\sigma_2 > \sigma_1$ but $< j_0$ if $\sigma_2 < \sigma_1$, as expected. Outside the circle, ($\rho > a$) and for $\cos\phi$ positive (positive x, i.e. to the right of the circle), j_ρ is positive if $\sigma_2 > \sigma_1$. Thus current lines converge inward to the circle for negative x and move back outward for positive x. Again this is what we expected.

1.2 Plotting the flow lines.

Remember that we can use a complex potential for 2-D problems (Lea §2.4, notes3half section 2), $\chi = \Phi + i\psi$, and the imaginary part $\psi =$ constant gives us the field lines. Here there is an extra subtlety because $\vec{j} = -\sigma\vec{\nabla}\Phi$, so $\sigma\psi$ gives the current flow lines. We have, with $r = \rho/a$, and $\frac{a}{\rho}\cos\phi = \mathrm{Re}\left[1/\left(re^{i\phi}\right)\right] = \mathrm{Re}\,(1/z)$

$$
\chi = \Phi + i\psi = -\frac{j_0 a}{\sigma_1}\begin{cases} \frac{2\sigma_1}{\sigma_2+\sigma_1} z & \text{if} \quad r < 1 \\ z - \frac{1}{z}\frac{(\sigma_2-\sigma_1)}{\sigma_2+\sigma_1} & \text{if} \quad r > 1 \end{cases} \tag{12}
$$

where

$$
\frac{1}{z} = \frac{1}{re^{i\phi}} = \frac{1}{r}e^{-i\phi} = \frac{1}{r}\left(\cos\phi - i\sin\phi\right)
$$

and thus, taking the imaginary part of (12), we have

$$
\sigma\psi = -j_0 a\begin{cases} \frac{2\sigma_2}{\sigma_2+\sigma_1} r\sin\phi & \text{if} \quad r < 1 \\ r\sin\phi\left(1 + \frac{1}{r^2}\frac{(\sigma_2-\sigma_1)}{\sigma_2+\sigma_1}\right) & \text{if} \quad r > 1 \end{cases}
$$

If $\sigma_2 \to 0$ no current flows through the circle and we retrieve the solution in Lea Ch2 §2.4.4 for fluid flow around a cylinder. If $\sigma_2 \to \sigma_1$, we retrieve the expected undeviated current flow. For $r < 1$,

$$
y_{\text{in}} = r\sin\phi = \frac{-\psi\sigma_2}{j_0 a}\frac{\sigma_2+\sigma_1}{2\sigma_2}
$$

So, since $|y_{\text{in}}| < 1$, values of

$$
k = \left|\frac{-\psi\sigma}{j_0 a}\right| < \frac{2\sigma_2}{\sigma_2+\sigma_1} = k_{\max}
$$

correspond to current flow lines that pass through the circle. The corresponding value outside the circle is

$$
y_{\text{out}} = r\sin\phi = \frac{k}{1+\frac{1}{r^2}\frac{(\sigma_2-\sigma_1)}{\sigma_2+\sigma_1}} = \frac{y_{\text{in}}}{1+\frac{1}{r^2}\frac{(\sigma_2-\sigma_1)}{\sigma_2+\sigma_1}}\frac{2\sigma_2}{\sigma_2+\sigma_1}
$$

Thus

$$\frac{y_{\text{out}}}{y_{\text{in}}} \to \frac{2\sigma_2}{\sigma_2+\sigma_1} \quad \text{as } r \to \infty$$

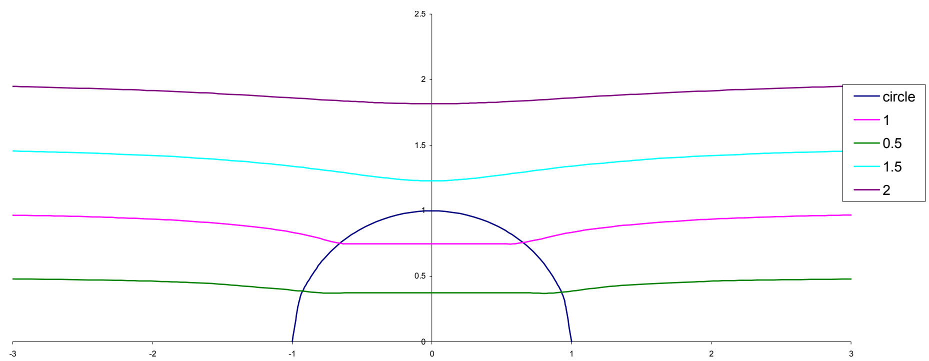
and this ratio is > 1 if $\sigma_2 > \sigma_1$, as expected.

Plot for $\sigma_2/\sigma_1 = 2$

$$k = \begin{cases} \frac{4}{3} r \sin\phi & if \quad r < 1 \\ r\sin\phi\left(1+\frac{1}{3r^2}\right) & if \quad r > 1 \end{cases}$$

Thus the flow lines are given by:

$$\begin{aligned} r(\phi) &= \frac{3}{4}\frac{k}{\sin\phi} && r < 1 \\ &= \frac{3k+\sqrt{9k^2-12\sin^2\phi}}{6\sin\phi} && r > 1 \end{aligned}$$



The line charge density at the circular boundary may be found from the normal component of E. If t is the thickness of the sheet, then

$$\begin{aligned} \left.\left(E_{\rho 1}-E_{\rho 2}\right)\right|_{\rho=a} &= \frac{\lambda}{\varepsilon_0 t} \\ &= \frac{j_0\cos\phi}{\sigma_1}+\frac{j_0}{\sigma_1}\frac{(\sigma_2-\sigma_1)}{\sigma_2+\sigma_1}\cos\phi-\frac{2j_0}{\sigma_2+\sigma_1}\cos\phi \\ &= \frac{j_0\cos\phi}{\sigma_1(\sigma_2+\sigma_1)}(\sigma_2+\sigma_1+\sigma_2-\sigma_1-2\sigma_1) \\ \lambda &= t\frac{2j_0\varepsilon_0\cos\phi}{\sigma_1}\left(\frac{\sigma_2-\sigma_1}{\sigma_2+\sigma_1}\right) \end{aligned}$$

Check the dimensions! λ is zero at the top and bottom of the cylinder where $\vec{j}$ is tangent to the circle, and is zero everywhere if $\sigma_2 = \sigma_1$. When the uniform field is turned on, it takes a very short time ($\sim a/c$) for the current flow to build up this charge at the boundary, at which point a steady state is achieved.