Propagation of EM Waves in material media

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1 Wave propagation

As usual, we start with Maxwell's equations with no free charges:

$$\vec{\nabla}\cdot\vec{D}=0$$
$$\vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}+\vec{j}$$

If we now assume that each field has the plane wave form $\vec{E}=\vec{E}_0e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ (or equivalently, we Fourier transform everything), the equations simplify:

$$\vec{k}\cdot\vec{D}=0 \tag{1}$$

$$\vec{k}\cdot\vec{B}=0 \tag{2}$$

$$\vec{k}\times\vec{E}=\omega\vec{B} \tag{3}$$

$$i\vec{k}\times\vec{H}=-i\omega\vec{D}+\vec{j} \tag{4}$$

Then, for an LIH, conducting medium, we can eliminate $\vec{D}=\varepsilon\vec{E}$, $\vec{H}=\vec{B}/\mu$ and $\vec{j}=\sigma\vec{E}$ to get:

$$\varepsilon\vec{k}\cdot\vec{E}=0$$

$$\vec{k}\times\vec{B}=-\omega\mu\varepsilon\vec{E}-i\mu\sigma\vec{E}=-\omega\mu\varepsilon\left(1+i\frac{\sigma}{\omega\varepsilon}\right)\vec{E} \tag{5}$$

For now let's take $\sigma=0$ (non-conducting medium). Then we get the wave equation:

$$\begin{aligned}\vec{k}\times\left(\vec{k}\times\vec{B}\right) &= \vec{k}\left(\vec{k}\cdot\vec{B}\right)-k^2\vec{B}=-\omega\mu\varepsilon\vec{k}\times\vec{E}\\ k^2\vec{B} &= \omega^2\mu\varepsilon\vec{B}\end{aligned} \tag{6}$$

where we used equation (2). So the wave phase speed is

$$v_\phi=\frac{\omega}{k}=\frac{1}{\sqrt{\mu\varepsilon}}$$

and the refractive index is:

$$n = \frac{c}{v_\phi} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \tag{7}$$

and

$$k = \frac{\omega}{c} n \tag{8}$$

Then equation (3) becomes:

$$n\hat{k} \times \vec{E} = c\vec{B} \tag{9}$$

For most LIH materials in which $\vec{B} = \mu\vec{H}$ is a useful relation, $\mu \simeq \mu_0$, so the refractive index is primarily determined by the dielectric constant $\varepsilon/\varepsilon_0$.

2 Reflection and transmission of waves at a boundary

Now let's consider a wave incident on a plane boundary between two media with refractive indices n_1 and n_2. We choose coordinates so that the boundary is the $x-y-$plane. The field in the incident wave is

$$\vec{E} = \vec{E}_i \exp\left(i\vec{k}_i \cdot \vec{x} - i\omega t\right) \tag{10}$$

In general there will be a transmitted wave with

$$\vec{E} = \vec{E}_t \exp\left(i\vec{k}_t \cdot \vec{x} - i\omega_t t\right) \tag{11}$$

and a reflected wave with

$$\vec{E} = \vec{E}_r \exp\left(i\vec{k}_r \cdot \vec{x} - i\omega_r t\right) \tag{12}$$

The **plane of incidence** is the plane containing the normal to the boundary and the incident ray, that is $\hat{n}$ and $\vec{k}_i$. We choose the axes so that the plane of incidence is the $x-z-$plane. The angle of incidence θ is the angle between $\vec{k}_i$ and $\hat{n} = \hat{z}$. (See Figure 1 in section 2.1.) Then

$$\vec{k}_i \cdot \vec{x} = k_i \left(x \sin\theta + z\cos\theta\right) \tag{13}$$

The boundary conditions we have to satisfy are (Notes 1 equations 10, 12, 13 and 15 with σ_f and $\vec{K}_f$ zero).

$$\hat{n} \cdot \vec{D} = D_z \text{ is continuous} \tag{14}$$

$$\hat{n} \cdot \vec{B} = B_z \text{ is continuous} \tag{15}$$

$$\vec{E}_{\tan} \text{ is continuous} \tag{16}$$

$$\vec{H}_{\tan} \text{ is continuous} \tag{17}$$

For most ordinary materials $\mu/\mu_0 \simeq 1$, so we shall ignore the difference between $\vec{B}$ and $\mu_0\vec{H}$.

In order to satisfy the boundary conditions at $z = 0$ for all times t and at all x and y, we must have (eqn 13) $\vec{k} \cdot \vec{x} - \omega t = kx\sin\theta - \omega t$ the same for each wave. That is, each wave has the same frequency ω and $k_x = k\sin\theta$ is the same for each wave. From (eqn 8) with fixed ω, we also have $k_r = k_i$ and $k_t = \frac{n_2}{n_1}k_i$. So the angle of incidence equals the angle of reflection, and

$$k_i \sin\theta = k_t \sin\theta_t = \frac{n_2}{n_1} k_i \sin\theta_t$$

or

$$n_1 \sin\theta = n_2 \sin\theta_t \tag{18}$$

which is Snell's law.

Notice that the physical argument that gives the laws of reflection and refraction (boundary conditions must hold for all time and everywhere on the boundary) is independent of the kind of wave and the specific form of the boundary conditions, and so these laws hold for waves of all kinds (sound waves, seismic waves, surface water waves, etc.).

We now have four equations to solve for the unknowns $\vec{E}_r$ and $\vec{E}_t$. However, two of the equations (16 and 17) are vector equations with two components, so we actually have six equations. Since we know the directions of the wave vectors $\vec{k}$, and $\vec{E}$ is perpendicular to $\vec{k}$, there are two components of $\vec{E}_r$ and $\vec{E}_t$ in the plane perpendicular to their respective $\vec{k}$, so we have four unknowns. Our equations are not all independent. We can simplify by decomposing the incident light into two specific linear polarizations.

2.1 Polarization perpendicular to the plane of incidence

In this polarization, $\vec{E}$ is perpendicular to the plane of incidence, (that is, with our chosen coordinates, $\vec{E} = E_y\hat{y}$). The vectors $\vec{E}$ and $\vec{B}$ in the waves are as shown in the diagram. The direction of $\vec{B}$ is chosen so that $\vec{S}$, the Poynting vector, is in the correct direction for each wave.

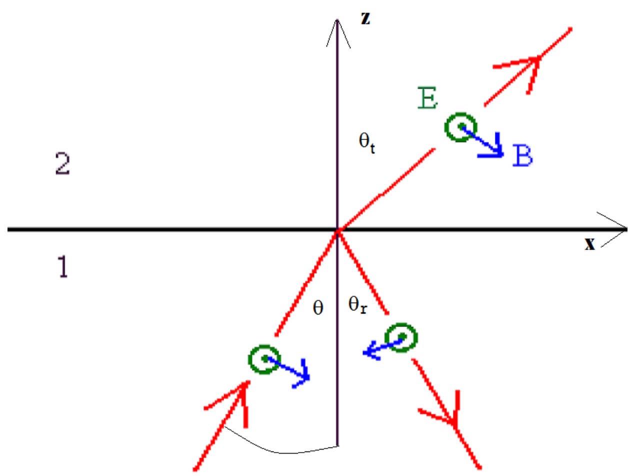

Figure 1

In this case there are only two unknowns, E_t and E_r, but the boundary condition (14) is trivially satisfied, since $D_z = 0$. The remaining conditions cannot all be independent. Eqn. (16) has only one non-zero component:

$$E_i + E_r = E_t \tag{19}$$

The exponentials in the expressions for the fields in eqns (14 - 17) cancel. Similarly from (17) we have:

$$B_{xi} + B_{xr} = B_{xt}$$

and from equation (9) we find $cB_x = -n\hat{k}_z E_y$, so we may rewrite this equation in terms of the electric field components.

$$n_1 (E_i - E_r) \cos\theta = n_2 E_t \cos\theta_t = n_2 (E_i + E_r) \cos\theta_t \tag{20}$$

where we used equation (19) in the last step. The final boundary condition is (15):

$$\begin{aligned} B_{zi} + B_{zr} &= B_{zt} \\ n_1 (E_i + E_r) \sin\theta &= n_2 E_t \sin\theta_t \end{aligned}$$

where we used $B_z = n\hat{k}_x E_y / c$. With Snell's law, this relation duplicates eqn (19), so we have two equations for two unknowns. Rearranging eqn (20), we get

$$E_i (n_1 \cos\theta - n_2 \cos\theta_t) = E_r (n_1 \cos\theta + n_2 \cos\theta_t)$$

Solving for E_r, and using Snell's law to eliminate θ_t, we have

$$\begin{aligned} E_r &= \frac{n_1 \cos\theta - n_2 \cos\theta_t}{n_1 \cos\theta + n_2 \cos\theta_t} E_i \\ &= \frac{n_1 \cos\theta - n_2 \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}}{n_1 \cos\theta + n_2 \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}} E_i \\ E_r &= \frac{n_1 \cos\theta - \sqrt{n_2^2 - n_1^2 \sin^2\theta}}{n_1 \cos\theta + \sqrt{n_2^2 - n_1^2 \sin^2\theta}} E_i \end{aligned} \tag{21}$$

The reflected amplitude depends on the angle of incidence, and on the ratio of the two refractive indices. From equation (21) we can conclude that E_r has the opposite sign from E_i if $n_2 > n_1$, independent of the angle θ. If $n_2 > n_1$,

$$\begin{aligned} 1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta &> 1 - \sin^2\theta = \cos^2\theta \\ n_2 \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta} &> n_1 \cos\theta \end{aligned}$$

Thus the reflected wave has a phase change of π if $n_2 > n_1$, as we learn in elementary optics.

Finally from equation (19), we find the transmitted amplitude:

$$E_t = E_i \frac{2n_1 \cos\theta}{n_1 \cos\theta + \sqrt{n_2^2 - n_1^2 \sin^2\theta}} \tag{22}$$

The time-averaged power transmitted is given by (eqn 9 and waveguide notes eqn 30)

$$< \vec{S} > \ = \frac{1}{2} \text{Re} \left(\vec{E} \times \vec{H}^* \right) = \frac{1}{2} \text{Re} \left[\vec{E} \times \left(\frac{n}{c\mu_0} \hat{k} \times \vec{E}^* \right) \right] = \hat{k} \frac{n}{2c\mu_0} |E|^2$$

Thus the sum of power transmitted normally across the boundary plus power reflected normally is

$$
\begin{aligned}
< \quad \vec{S}_r \cdot \hat{n} > + < \vec{S}_t \cdot \hat{n} > \quad &= \frac{n_1}{2c\mu_0} |E_r|^2 \cos\theta + \frac{n_2}{2c\mu_0} |E_t|^2 \cos\theta_t \\
&= \frac{|E_i|^2}{2c\mu_0} \left\{ n_1 \left[\frac{n_1 \cos\theta - \sqrt{n_2^2 - n_1^2 \sin^2\theta}}{n_1 \cos\theta + \sqrt{n_2^2 - n_1^2 \sin^2\theta}} \right]^2 \cos\theta \right. \\
&\quad \left. + n_2 \left[\frac{2 n_1 \cos\theta}{n_1 \cos\theta + \sqrt{n_2^2 - n_1^2 \sin^2\theta}} \right]^2 \frac{\sqrt{n_2^2 - n_1^2 \sin^2\theta}}{n_2} \right\} \\
&= \frac{|E_i|^2}{2c\mu_0} n_1 \cos\theta \frac{n_1^2 \cos^2\theta + 2 n_1 \cos\theta \sqrt{n_2^2 - n_1^2 \sin^2\theta} + n_2^2 - n_1^2 \sin^2\theta}{\left(n_1 \cos\theta + \sqrt{n_2^2 - n_1^2 \sin^2\theta} \right)^2} \\
&= \frac{|E_i|^2}{2c\mu_0} n_1 \cos\theta = \quad < \vec{S}_i \cdot \hat{n} > \quad \text{as expected!}
\end{aligned}
$$

2.2 Polarization parallel to the plane of incidence

The $\vec{E}$ and $\vec{B}$ fields in this polarization look like this:

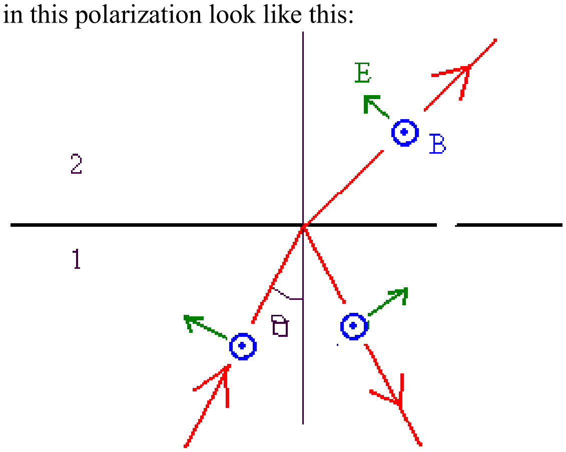

Figure 2

and boundary condition (15) is trivially satisfied. Boundary condition (17) with $\mu_1 = \mu_2 = \mu_0$ together with eqn (9) gives

$$
\begin{aligned}
B_i + B_r &= B_t \\
n_1 (E_i + E_r) &= n_2 E_t
\end{aligned} \tag{23}
$$

Then (16) together with Snells' law gives:

$$
(E_i - E_r) \cos\theta = E_t \cos\theta_t = \frac{n_1}{n_2} (E_i + E_r) \sqrt{1 - \left(\frac{n_1}{n_2} \right)^2 \sin^2\theta} \tag{24}
$$

The final boundary condition (14) is

$$\varepsilon_1 \left(E_i + E_r\right) \sin\theta = \varepsilon_2 E_t \sin\theta_t$$

and, since $\varepsilon_1 \propto n_1^2$ (eqn 7), this duplicates eqn (23).

Eqn (24) gives E_r:

$$E_i \left(\cos\theta - \frac{n_1}{n_2}\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}\right) = E_r \left(\cos\theta + \frac{n_1}{n_2}\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}\right)$$

So

$$\begin{aligned} E_r &= E_i \frac{\cos\theta - \frac{n_1}{n_2}\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}}{\cos\theta + \frac{n_1}{n_2}\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}} \\ &= E_i \frac{n_2^2 \cos\theta - n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta}}{n_2^2 \cos\theta + n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta}} \end{aligned} \tag{25}$$

and then from (23):

$$\begin{aligned} E_t &= \frac{n_1}{n_2} E_i \frac{2n_2^2 \cos\theta}{n_2^2 \cos\theta + n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta}} \\ &= E_i \frac{2n_1 n_2 \cos\theta}{n_2^2 \cos\theta + n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta}} \end{aligned} \tag{26}$$

2.3 Polarization by reflection

Since the reflected amplitudes (25) and (21) are not the same in the two different polarizations, the reflected light is always partially polarized when the incident light is unpolarized. Equation (25) shows that the reflected amplitude in the polarization parallel to the plane of incidence is zero if:

$$n_2^2 \cos\theta - n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta} = 0$$

or, squaring and writing $1 = \cos^2\theta + \sin^2\theta$, we have

$$\begin{aligned} n_2^4 \cos^2\theta &= n_1^2 \left[n_2^2 \left(\cos^2\theta + \sin^2\theta\right) - n_1^2 \sin^2\theta\right] \\ n_2^2 \cos^2\theta \left(n_2^2 - n_1^2\right) &= n_1^2 \sin^2\theta \left(n_2^2 - n_1^2\right) \end{aligned}$$

Thus either $n_1 = n_2$ (no boundary), or

$$\tan\theta = \frac{n_2}{n_1} \tag{27}$$

This is Brewster's angle.

Can this also happen for the other polarization? We would need:

$$\begin{aligned} n_1 \cos\theta - \sqrt{n_2^2 - n_1^2 \sin^2\theta} &= 0 \\ n_1^2 \cos^2\theta &= n_2^2 - n_1^2 \sin^2\theta \end{aligned}$$

or $n_2 = n_1$. So the reflected field amplitude for this polarization is not zero unless there is no boundary. Thus when unpolarized light is incident at Brewster's angle, the reflected light is 100% polarized perpendicular to the plane of incidence. At other angles of incidence, the reflected light is partially polarized.

Why does this happen? At Brewster's angle,

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{\sin\theta_i}{\tan\theta_i} = \cos\theta_i$$

Thus

$$\theta_t = \frac{\pi}{2} - \theta_i$$

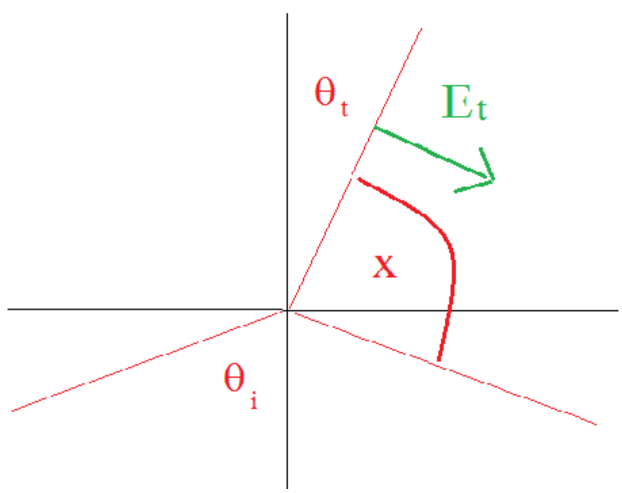

Figure 3

The angle between the reflected and transmitted waves is

$$\chi = \frac{\pi}{2} - \theta_t + \frac{\pi}{2} - \theta_i = \frac{\pi}{2} - \theta_t + \theta_t = \frac{\pi}{2}$$

Thus electrons accelerated by the electric field in medium 2 would need to radiate along the direction of the acceleration in order to create the reflected wave. This is impossible. (wavemks notes, see eg eqn 38.)

3 Waves in a Conducting medium

3.1 Propagation

If the medium is a conductor, (non-zero σ in eqn (5)), the dispersion relation (6) becomes:

$$k^2 = \omega^2 \mu\varepsilon \left(1 + i\frac{\sigma}{\omega\varepsilon}\right) \tag{28}$$

and thus the solution for k is complex.

$$k = |k|\, e^{i\phi} = |k| \left(\cos\phi + i\sin\phi\right) = \kappa + i\gamma$$

where

$$|k|^2 = \omega^2 \mu\varepsilon \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} \tag{29}$$

$$\tan 2\phi = \frac{\sigma}{\omega\varepsilon} \tag{30}$$

Then

$$e^{i\vec{k}\cdot\vec{x}} = e^{i\kappa\hat{k}\cdot\vec{x}} e^{-\gamma\hat{k}\cdot\vec{x}}$$

The imaginary part γ of k indicates spatial attenuation of the wave,

$$\operatorname{Im} k = \gamma = \omega\sqrt{\mu\varepsilon}\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4} \sin\phi$$

while the real part κ of k gives the wave phase speed

$$v_{\text{ph}} = \omega/\kappa.$$

If σ is small ($\sigma/\omega\varepsilon \ll 1$), we may expand the functions (29) and (30) to first order:

$$|k| = \omega\sqrt{\mu\varepsilon}\left(1 + \frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right) \simeq \omega\sqrt{\mu\varepsilon}$$

$$\phi = \frac{\sigma}{2\omega\varepsilon}$$

and we get back the expected results as $\sigma \to 0$. The imaginary part of k is small because ϕ is small.

$$\operatorname{Im}(k) \simeq \omega\sqrt{\mu\varepsilon}\frac{\sigma}{2\omega\varepsilon} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

and the wave phase speed is almost unchanged:

$$v_{\text{ph}} \simeq \frac{1}{\sqrt{\mu\varepsilon}}$$

But if σ is large ($\sigma/\omega\varepsilon \gg 1$), $\phi \to \pi/4$, and

$$|k| \simeq \omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{\omega\varepsilon}} = \sqrt{\mu\sigma\omega}$$

$$\operatorname{Im}(k) \simeq \sqrt{\mu\sigma\omega}\frac{1}{\sqrt{2}} = \sqrt{\frac{\mu\sigma\omega}{2}} = \operatorname{Re}(k)$$

The wave is damped within a short distance

$$\delta \sim 1/\gamma = \sqrt{\frac{2}{\sigma\mu\omega}} \tag{31}$$

The distance δ is called the skin depth.

In a conducting medium the relations between the fields are (eqn 3):

$$\vec{B} = \frac{|k|}{\omega} e^{i\phi} \left(\hat{k} \times \vec{E}\right),$$

so ϕ is also the phase shift between the fields. If γ is small ($\gamma \ll \kappa$, low conductivity) then the amplitude of $\vec{B}$ is almost the same[1] as the amplitude of $\vec{E}$ times $\sqrt{\mu\varepsilon}$, and the phase shift is also small. But if γ is large ($\kappa \simeq \gamma$, high conductivity $\sigma/\omega\varepsilon \gg 1$) then $\left|\vec{B}\right|$ is much larger than $\sqrt{\mu\varepsilon}\left|\vec{E}\right|$, and the phase shift is almost $\pi/4$. Also the phase speed is given by

$$\frac{v_\phi}{c} = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\mathrm{Re}\,(k)} = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\sqrt{\frac{\mu\sigma\omega}{2}}} = \sqrt{\frac{2\omega\varepsilon_0}{\sigma}} \ll 1$$

To summarize, in a good conductor the wave fields are primarily magnetic, $\vec{E}$ and $\vec{B}$ are out of phase by $\pi/4$, and the wave phase speed is very slow.

3.2 Reflection and refraction at a boundary with a conducting medium

How does this affect the reflection and refraction of a wave? Consider a wave with wave number k incident on a conducting medium. Inside the conductor the fields behave like

$$\exp\left(ik_t x \sin\theta_t + ik_t z\cos\theta_t - i\omega t\right) = \exp\left(ikx\sin\theta + ik_t z\sqrt{1 - \frac{k^2}{k_t^2}\sin^2\theta} - i\omega t\right)$$

where k is the incident wave number. For a good conductor, with

$$\kappa \simeq \gamma = \sqrt{\frac{\sigma\mu\omega}{2}} = \frac{1}{\delta}$$

we have

$$k\delta = \omega\sqrt{\mu_0\varepsilon_1}\sqrt{\frac{2}{\sigma\mu\omega}} \simeq \sqrt{\frac{2\varepsilon_1\omega}{\sigma}} = \sqrt{2}\frac{n_1}{n_2}\left(\frac{\sigma}{\omega\varepsilon_2}\right)^{-1/2}$$

So $k\delta$ is small when $\sigma/\omega\varepsilon_2$ is large. Then the coefficient of z is

$$ik_t\cos\theta_t = i\sqrt{k_t^2 - k^2\sin^2\theta} = i\sqrt{\kappa^2 - \gamma^2 - k^2\sin^2\theta + 2i\gamma\kappa} = i\sqrt{-k^2\sin^2\theta + 2i\gamma\kappa}$$

[1] In Gaussian units, $B \simeq E$.

or

$$\begin{aligned} ik_t\cos\theta_t &= -k\sin\theta\sqrt{1-i\frac{2}{k^2\delta^2\sin^2\theta}} \\ &= -k\sin\theta\left[1+\left(\frac{2}{k^2\delta^2\sin^2\theta}\right)^2\right]^{1/4} e^{-i\chi/2} \\ &\simeq -k\frac{\sqrt{2}}{k\delta}e^{-i\chi/2} = -\frac{\sqrt{2}}{\delta}e^{-i\chi/2} \qquad (k\delta \ll 1) \end{aligned} \tag{32}$$

where

$$\tan\chi = \frac{2}{k^2\delta^2\sin^2\theta}$$

and thus $\chi \simeq \pi/2$, and $e^{-i\chi/2} \simeq (1-i)/\sqrt{2}$. Thus the field components in the conductor are proportional to

$$\exp\left[-\frac{z}{\delta}(1-i)\right]$$

This shows that the fields in the conducting medium propagate only a distance of order δ in the $z-$direction (normal to the boundary).

For polarization perpendicular to the plane of incidence (only E_y non-zero), equation (19) still holds, but the second boundary condition (equation 17) becomes (using 32):

$$\begin{aligned} (E_i - E_r)\cos\theta &= E_t\frac{k_t}{k}\cos\theta_t = E_t i\frac{\sqrt{2}}{k\delta}e^{-i\chi/2} \\ &\simeq E_t\frac{(1+i)}{k\delta} \end{aligned}$$

Combining with equation (19), we have:

$$2E_i = E_t\left(1+\frac{i+1}{k\delta\cos\theta}\right) \simeq E_t\frac{1}{k\delta}\left(\frac{i+1}{\cos\theta}\right)$$

or

$$E_t = 2E_i\frac{k\delta\cos\theta}{(1+i)} = E_i k\delta\cos\theta\ (1-i)$$

which is very small ($|E_t| \ll E_i$). Thus almost all the wave energy is reflected. A similar result holds for the other polarization. (See also Jackson problems 7.4 and 7.5.)