Abraham-Lorentz Self -force

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Abraham and Lorentz (circa 1904) attempted a rigorous derivation fo the radiation reaction force. The idea is to model a particle as a collection of charges, currents and fields. The basic equation of motion is:

$$\frac{d\vec{P}_{\text{particle}}}{dt} = \vec{F}_{\text{ext}} \tag{1}$$

where $\vec{F}_{\text{ext}}$ is the sum of the *external* forces. We assume for the moment that *all* the forces acting are electromagnetic in origin. Then the system's total momentum is the sum of its (apparently) mechanical momentum and the momentum of the fields. Thus:

$$\frac{d\vec{P}_{\text{mech}}}{dt} + \frac{d\vec{P}_{\text{fields}}}{dt} = 0$$

The rate of change of field momentum may be expressed in terms of the Lorentz force density:

$$\frac{d\vec{P}_{\text{fields}}}{dt} = -\int \left(\rho\tilde{\mathbf{E}}+\frac{1}{c}\tilde{\mathbf{j}} \times \tilde{\mathbf{B}}\right) dV$$

(See eg equation 12.120). We decompose the fields into the sum of the self fields plus the external fields:

$$\vec{E} = \vec{E}_s + \vec{E}_{\text{ext}}$$

Thus:

$$\begin{aligned}
\int \left(\rho\vec{E} + \frac{1}{c}\vec{j} \times \vec{B}\right) dV &= \int \left(\rho\vec{E}_{\text{ext}} + \frac{1}{c}\vec{j} \times \vec{B}_{\text{ext}}\right) dV + \int \left(\rho\vec{E}_s + \frac{1}{c}\vec{j} \times \vec{B}_s\right) dV \\
&= \vec{F}_{\text{ext}} + \int \left(\rho\vec{E}_s + \frac{1}{c}\vec{j} \times \vec{B}_s\right) dV
\end{aligned}$$

Thus we have:

$$\frac{d\vec{P}_{\text{mech}}}{dt} = -\frac{d\vec{P}_{\text{fields}}}{dt} = \vec{F}_{\text{ext}} + \int \left(\rho\vec{E}_s + \frac{1}{c}\vec{j} \times \vec{B}_s\right) dV \tag{2}$$

where the second term on the right hand side is the self force.

To calculate this integral we assume:

- The particle is instantaneously at rest
- the charge distribution is rigid and spherically symmetric

Then the expression for the self-force simplifies, since $\vec{j} = \rho\vec{v} = 0$:

$$\begin{aligned} \vec{F}_{\text{self}} &= \int \rho \vec{E}_s dV \\ &= -\int \rho \left(\vec{\nabla}\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right) dV \end{aligned}$$

where

$$\Phi = \int \frac{\rho\left(\vec{x}', t'\right)\big|_{\text{ret}}}{R} dV'$$

and similarly for $\vec{A}$:

$$\vec{A} = \frac{1}{c}\int \frac{\vec{j}\left(\vec{x}', t'\right)\Big|_{\text{ret}}}{R} dV'$$

The retarded time $t' = t - R/c$ differs from t by a term of order a/c where a is the radius of the particle. This time is extremely short, and the particle cannot move far during this time interval. Thus we may expand

$$\begin{aligned} \rho\left(\vec{x}', t'\right)\big|_{\text{ret}} &= \rho\left(\vec{x}', t\right) - \frac{R}{c}\frac{\partial \rho}{\partial t} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{R}{c}\right)^n \frac{\partial^n \rho}{\partial t^n} \end{aligned}$$

and thus:

$$\begin{aligned} \int \rho \left(\vec{\nabla}\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right) dV &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^n} \int \rho\left(\vec{x},t\right) \vec{\nabla} R^{n-1} \frac{\partial^n \rho\left(\vec{x}', t\right)}{\partial t^n} + \frac{\rho\left(\vec{x},t\right)}{c^2} \frac{\partial}{\partial t} R^{n-1} \frac{\partial^n \vec{j}\left(\vec{x}', t\right)}{\partial t^n} d^3x' d^3x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^n} \int \rho\left(\vec{x}, t\right) \frac{\partial^n}{\partial t^n} \left[\rho\left(\vec{x}', t\right) \vec{\nabla} R^{n-1} + \frac{R^{n-1}}{c^2} \frac{\partial \vec{j}\left(\vec{x}', t\right)}{\partial t} \right] d^3x' d^3x \end{aligned}$$

Now we can evaluate this expression term by term.

First the scalar potential part:

$n = 0$:

$$\int \rho\left(\vec{x},t\right) \rho\left(\vec{x}', t\right) \vec{\nabla}\frac{1}{R} d^3x' d^3x = -\int \rho\left(\vec{x},t\right) \rho\left(\vec{x}', t\right) \frac{\vec{x} - \vec{x}'}{R^3} d^3x' d^3x = 0$$

by symmetry. (Interchange $\vec{x}$ and $\vec{x}'$. The integral should not change, $I_1 = I_2$, but the integrand changes sign, implying $I_1 = -I_2$. So $I = 0$.)

(In detail...For a spherically symmetric distribution, $\rho\left(\vec{x},t\right) = \rho\left(r,t\right)$. Write $1/R$ as an expansion in Legendre polynomials. Put the polar axis along $\vec{x}$ while we do the x' integration:

$$\int d^3x \rho\left(r,t\right) \vec{\nabla} \int \rho\left(r',t\right) \sum_l \frac{r_<^l}{r_>^{l+1}} P_l\left(\mu'\right) r'^2 dr' d\mu' d\phi'$$

The integral over μ' is zero unless $l = 0$. So we have:

$$\begin{aligned} \int d^3x \rho\left(r,t\right) \vec{\nabla} \int \rho\left(r',t\right) \frac{1}{r_>} r'^2 dr' 4\pi &= 4\pi \int d^3x \rho\left(r,t\right) \vec{\nabla} \left(\int_0^r \rho\left(r',t\right) \frac{1}{r} r'^2 dr' + \int_r^a \rho\left(r',t\right) \frac{1}{r'} r'^2 dr' \right) \\ &= 4\pi \int d^3x \rho\left(r,t\right) \vec{\nabla} f\left(r\right) \end{aligned}$$

where $q(r)$ is the charge inside radius r . The right hand side is then:

$$4\pi\int d^3x\rho(r,t)\,\hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{q(r)}{r}+\int_r^a\rho(r',t)\,r'dr'\right)=4\pi\int d^3x\rho(r,t)\,\hat{\mathbf{r}}\left(-\frac{q(r)}{r^2}+\right)$$

Now $\hat{\mathbf{r}}=\hat{\mathbf{z}}\mu+\sin\theta\left(\hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi\right)$,so integrating over ϕ reduces the $x-$ and $y-$components to zero, while integrating over μ reduces the $\hat{\mathbf{z}}$ component to zero.)

$n=1$: $R^{n-1}\equiv 1$ and $\vec{\nabla}1=0$,so this term is zero too.

$n>1$: Relabel by setting $m=n-2$

Now we want to combine the remaining $(n>1)$ terms with the vector potential terms, so let's relabel the ρ terms by setting $n=m+2$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int\rho(\vec{x},t)\frac{\partial^n}{\partial t^n}\left[\rho(\vec{x}',t)\,\vec{\nabla}R^{n-1}+\frac{R^{n-1}}{c^2}\frac{\partial\vec{j}(\vec{x}',t)}{\partial t}\right]d^3x'd^3x\\
=&\sum_{n=2}^{\infty}\frac{(-1)^n}{n!c^n}\int\rho(\vec{x},t)\frac{\partial^n}{\partial t^n}\rho(\vec{x}',t)\,\vec{\nabla}R^{n-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int\rho(\vec{x},t)\frac{\partial^n}{\partial t^n}\frac{R^{n-1}}{c^2}\frac{\partial\vec{j}(\vec{x}',t)}{\partial t}d^3x'd^3x\\
=&\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+2)!c^{m+2}}\int\rho(\vec{x},t)\frac{\partial^{m+2}}{\partial t^{m+2}}\rho(\vec{x}',t)\,\vec{\nabla}R^{m+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int\rho(\vec{x},t)\frac{\partial^n}{\partial t^n}\frac{R^{n-1}}{c^2}\frac{\partial\vec{j}(\vec{x}',t)}{\partial t}d^3x'd^3x\\
=&\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int\rho(\vec{x},t)\,R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\left[\frac{\partial\rho(\vec{x}',t)}{\partial t}\frac{\vec{\nabla}R^{n+1}}{(n+2)(n+1)R^{n-1}}+\vec{j}(\vec{x}',t)\right]d^3x'd^3x
\end{aligned}$$

Now use charge continuity to write

$$\frac{\partial\rho(\vec{x}',t)}{\partial t}=-\vec{\nabla}'\cdot\vec{j}(\vec{x}',t)$$

Our expression becomes:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int\rho(\vec{x},t)\,R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\left[-\vec{\nabla}'\cdot\vec{j}(\vec{x}',t)\frac{(n+1)\,R^n\hat{\mathbf{R}}}{(n+2)(n+1)R^{n-1}}+\vec{j}(\vec{x}',t)\right]d^3x'd^3x\\
=&\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int\rho(\vec{x},t)\,R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\left[\vec{j}(\vec{x}',t)-\vec{\nabla}'\cdot\vec{j}(\vec{x}',t)\frac{\vec{R}}{(n+2)}\right]d^3x'd^3x \qquad (3)
\end{aligned}$$

Now integrate the last term over the prime variables. We use the usual bag of tricks:

$$\vec{\nabla}'\times\left(\vec{j}\times\vec{R}R^{n-1}\right)=\vec{j}\left(\vec{\nabla}'\cdot\vec{R}R^{n-1}\right)-R^{n-1}\vec{R}\left(\vec{\nabla}'\cdot\vec{j}\right)+R^{n-1}\left(\vec{R}\cdot\vec{\nabla}'\right)\vec{j}-\left(\vec{j}\cdot\vec{\nabla}'\right)\vec{R}R^{n-1}$$

Thus:

$$R^{n-1}\vec{R}\left(\vec{\nabla}'\cdot\vec{j}\right)=-\vec{\nabla}'\times\left(\vec{j}\times\vec{R}R^{n-1}\right)+\vec{j}\left(\vec{\nabla}'\cdot\vec{R}R^{n-1}\right)+R^{n-1}\left(\vec{R}\cdot\vec{\nabla}'\right)\vec{j}-\left(\vec{j}\cdot\vec{\nabla}'\right)\vec{R}R^{n-1}$$

The integral of the curl converts to

$$\int_S\hat{\mathbf{n}}'\times\left(\vec{j}\times\vec{R}R^{n-1}\right)d^2x'=0$$

and

$$\begin{aligned}
\vec{j}\left(\vec{\nabla}'\cdot\vec{R}R^{n-1}\right) &= -\vec{j}\left(\vec{\nabla}\cdot\vec{R}R^{n-1}\right)\\
&= \vec{\nabla}\times\left(\vec{R}R^{n-1}\times\vec{j}\right)-R^{n-1}\vec{R}\left(\vec{\nabla}\cdot\vec{j}\right)-\left(\vec{j}\cdot\vec{\nabla}\right)\vec{R}R^{n-1}+R^{n-1}\left(\vec{R}\cdot\vec{\nabla}\right)\vec{j}\\
&= (\text{integrates to zero})-0-\left(\vec{j}\cdot\vec{\nabla}\right)\vec{R}R^{n-1}+0\\
&= \left(\vec{j}\cdot\vec{\nabla}'\right)\vec{R}R^{n-1}
\end{aligned}$$

Also:

$$R^{n-1}\left(\vec{R}\cdot\vec{\nabla}'\right)\vec{j}=\vec{\nabla}'\left(R^{n-1}\vec{R}\cdot\vec{j}\right)-\left(\vec{j}\cdot\vec{\nabla}'\right)R^{n-1}\vec{R}-R^{n-1}\vec{R}\times\left(\vec{\nabla}'\times\vec{j}\right)-\vec{j}\times\left(\vec{\nabla}'\times\vec{R}R^{n-1}\right)$$

The first term converts to a surface integral via a divergence theorem variant (see cover of J), and the integral is zero. The last term is zero since $\vec{\nabla}'\times\vec{R}=0$. Since our object is rigid, $\vec{j}$ can either be in a constant direction, or due to rotation of the sphere. Jackson does not seem to allow for rotation, in which case

$$\vec{\nabla}'\times\vec{j}=\vec{\nabla}'\times\left[\rho\left(r'\right)\vec{v}\right]=\frac{\partial\rho}{\partial r'}\hat{\mathbf{r}}'\times\vec{v}+\rho\vec{\nabla}'\times\vec{v}=\frac{\partial\rho}{\partial r'}\hat{\mathbf{r}}'\times\vec{v}$$

Then

$$\int R^{n-1}\vec{R}\times\left(\vec{\nabla}'\times\vec{j}\right)d^3x'=\int R^{n-1}\vec{R}\times\left(\frac{\partial\rho}{\partial r'}\hat{\mathbf{r}}'\times\vec{v}\right)d^3x'$$

If we choose polar axis along $\vec{v}$,then the right hand side is:

$$-\int R^{n-1}\vec{R}\times\left(\frac{\partial\rho}{\partial r'}\hat{\boldsymbol{\phi}}\sin\theta\right)d^3x'=0$$

Thus:

$$R^{n-1}\left(\vec{R}\cdot\vec{\nabla}'\right)\vec{j}=-\left(\vec{j}\cdot\vec{\nabla}'\right)R^{n-1}\vec{R}$$

and thus:

$$\begin{aligned}
\int R^{n-1}\vec{R}\left(\vec{\nabla}'\cdot\vec{j}\right)d^3x' &= \int\left[\left(\vec{j}\cdot\vec{\nabla}'\right)\vec{R}R^{n-1}-\left(\vec{j}\cdot\vec{\nabla}'\right)R^{n-1}\vec{R}-\left(\vec{j}\cdot\vec{\nabla}'\right)\vec{R}R^{n-1}\right]d^3x'\\
&= -\int\left(\vec{j}\cdot\vec{\nabla}'\right)\vec{R}R^{n-1}d^3x'
\end{aligned}$$

and finally!

$$\begin{aligned}
-\int R^{n-1}\vec{\nabla}'\cdot\vec{j}\left(\vec{x}',t\right)\frac{\vec{R}}{(n+2)}d^3x' &= \int\left(\vec{j}\left(\vec{x}',t\right)\cdot\vec{\nabla}'\right)\frac{R^{n-1}\vec{R}}{(n+2)}d^3x'\\
&= \frac{1}{n+2}\int R^{n-1}\vec{j}+(n-1)\,\vec{j}\cdot\hat{\mathbf{R}}R^{n-2}\vec{R}d^3x'
\end{aligned}$$

and so from equation (3) we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int\rho\left(\vec{x},t\right)R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\left[\vec{j}\left(\vec{x}',t\right)\frac{n+1}{n+2}-\frac{n-1}{n+2}\left(\vec{j}\left(\vec{x}',t\right)\cdot\hat{\mathbf{R}}\right)\hat{\mathbf{R}}\right]d^3x'd^3x$$

Now we set $\vec{j} = \rho\vec{v}$ (rigid assumption):

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^{n+2}} \int \rho\left(\vec{x}, t\right) R^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \rho \left[\vec{v}\left(t\right) \frac{n+1}{n+2} - \frac{n-1}{n+2} \left(\vec{v}\left(t\right) \cdot \hat{\mathbf{R}}\right) \hat{\mathbf{R}}\right] d^3x' d^3x$$

Next choose polar axis along $\tilde{\mathbf{v}}$, and the integral becomes:

$$I_n = \int \rho\left(\vec{x}, t\right) R^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \rho \left[\vec{v}\left(t\right) \frac{n+1}{n+2} - \frac{n-1}{n+2} \vec{v} \cos^2\gamma - \frac{n-1}{n+2} v \cos\gamma \hat{\perp}\right] d^3x' d^3R$$

Because of the symmetry, the direction of $\vec{v}$ is the only direction singled out, and so the $\perp-$component must integrate to zero. (It is not trivial to demonstrate this explicitly.) Then:

$$I_n = \int \rho\left(\vec{x}\right) R^{n-1} \frac{\partial^{n+1}\vec{v}\left(t\right)}{\partial t^{n+1}} \rho \left[\frac{n+1}{n+2} - \frac{n-1}{n+2} \cos^2\gamma\right] d^3x' R^2 dR d\mu$$

Term by term: The easiest term is:

$n = 1$

$$\begin{aligned} I_1 &= \int \rho\left(\vec{x}, t\right) \frac{\partial^2}{\partial t^2} \rho \vec{v}\left(t\right) \frac{2}{3} d^3x' d^3R = \frac{2}{3} \frac{\partial^2 \vec{v}\left(t\right)}{\partial t^2} \int \rho\left(\vec{x}\right) d^3x \rho\left(\vec{x}'\right) d^3x' \\ &= \frac{2}{3} \frac{\partial^2 \vec{v}\left(t\right)}{\partial t^2} e^2 \end{aligned}$$

and thus the $n = 1$ term in our series is:

$$-\frac{2}{3} \frac{e^2}{c^3} \frac{\partial^2 \vec{v}\left(t\right)}{\partial t^2}$$

and this contributes a term $F_{self} = \frac{2}{3}\frac{e^2}{c^3}\frac{\partial^2 \vec{v}(t)}{\partial t^2}$ which is exactly the expression we obtained before for the radiation damping self-force

$n = 0$ To do the prime integral, put $\vec{x}$ in the $x - z$ plane. Then $\phi = 0$:

$$I_0 = \frac{1}{2} \frac{\partial \vec{v}}{\partial t} \int \frac{\rho\left(\vec{x}, t\right)}{R} \rho \left(1 + \cos^2\gamma\right) d^3x' d^3x$$

where γ is the angle between $\vec{R}$ and $\vec{v}$. Now here Jackson inserts the average value of $\cos^2\gamma = 1/3$, which I think is incorrect (or at least needs better justification) to obtain for the $n = 0$ term:

$$\begin{aligned} F_0 &= \frac{\vec{a}}{c^2} \frac{4}{3} \frac{1}{2} \int \frac{\rho\left(r\right) \rho\left(r'\right)}{R} d^3x' d^3x \\ &= \frac{\vec{a}}{c^2} \frac{4}{3} U \end{aligned}$$

where U is the electromagnetic energy of the sphere. Then if we interpret $U/c^2 = m$, the mass of the particle, we get:

$$F_0 = \frac{4}{3} m\vec{a}$$

Thus we have , including only the first 2 terms in the series,

$$\frac{4}{3}m\vec{a}-\frac{2}{3}\frac{e^2}{c^3}\frac{\partial^2\vec{v}\,(t)}{\partial t^2}=\vec{F}_{\text{ext}}$$

In an alternative approach, we take the Fourier transformof equation (1) or equivalently (2) which now reads:

$$\frac{d\vec{P}_{\text{mech}}}{dt}=\vec{F}_{\text{ext}}-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int\rho\left(\vec{x}\right)R^{n-1}\frac{\partial^{n+1}\vec{v}\,(t)}{\partial t^{n+1}}\rho\left[\frac{n+1}{n+2}-\frac{n-1}{n+2}\cos^2\gamma\right]d^3x'd^3x$$

Taking the FT we get:

$$-i\omega m_0\vec{v}\,(\omega)=\vec{F}_{\text{ext}}\,(\omega)-\sum_{n=0}^{\infty}\frac{(-1)^n\,(-i\omega)^{n+1}}{n!c^{n+2}}\int\rho\left(\vec{x}\right)R^{n-1}\vec{v}\,(\omega)\,\rho\left(\vec{x}'\right)\left[\frac{n+1}{n+2}-\frac{n-1}{n+2}\cos^2\gamma\right]d^3x'd^3x$$

and taking the average of $\cos^2\gamma$ per JDJ we have

$$-i\omega m_0\vec{v}\,(\omega)=\vec{F}_{\text{ext}}\,(\omega)+\frac{2}{3}i\omega\vec{v}\,(\omega)\sum_{n=0}^{\infty}\frac{(i\omega)^n}{n!c^{n+2}}\int\rho\left(\vec{x}\right)R^{n-1}\rho\left(\vec{x}'\right)d^3x'd^3x$$

We can rewrite this expression as:

$$-i\omega M\left(\omega\right)\vec{v}\,(\omega)=\vec{F}_{\text{ext}}\,(\omega)$$

where

$$\begin{aligned}
M\left(\omega\right) &= m_0+\frac{2}{3c^2}\sum_{n=0}^{\infty}\frac{(i\omega)^n}{n!c^n}\int\rho\left(\vec{x}\right)R^{n-1}\rho\left(\vec{x}'\right)d^3x'd^3x\\
&= m_0+\frac{2}{3c^2}\int\rho\left(\vec{x}\right)\sum_{n=0}^{\infty}\frac{(i\omega R)^n}{n!c^n}\frac{\rho\left(\vec{x}'\right)}{R}d^3x'd^3x\\
M\left(\omega\right) &= m_0+\frac{2}{3c^2}\int\rho\left(\vec{x}\right)\frac{\rho\left(\vec{x}'\right)}{R}e^{i\omega R/c}d^3x'd^3x
\end{aligned}$$

Now the form factor for a particle is defined as the Fourier transform of ρ :

$$\rho\left(\vec{x}\right)=\frac{e}{(2\pi)^3}\int f\left(\vec{k}\right)e^{ik\cdot\vec{x}}d^3\vec{k}$$

So we may write:

$$\begin{aligned}
\int\rho\left(\vec{x}\right)\frac{\rho\left(\vec{x}'\right)}{R}e^{i\omega R/c}d^3x'd^3x &= \int\int\frac{e^2}{(2\pi)^3}\int f\left(\vec{k}\right)e^{ik\cdot\vec{x}}d^3\vec{k}\frac{1}{(2\pi)^3}\int f\left(\vec{k}'\right)e^{ik'\cdot\vec{x}'}d^3\vec{k}'\frac{e^{i\omega R/c}}{R}d^3x'd^3x\\
&= \int d^3\vec{k}\int d^3\vec{k}'\frac{e^2}{(2\pi)^6}f\left(\vec{k}\right)f\left(\vec{k}'\right)\int\int e^{ik\cdot\vec{x}}e^{ik'\cdot\vec{x}'}\frac{e^{i\omega R/c}}{R}d^3x'd^3x
\end{aligned}$$

Change variables to $\vec{R} = \vec{x} - \vec{x}'$

$$\begin{aligned}
\int\int e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'} \frac{e^{i\omega R/c}}{R} d^3x' d^3x &= \int\int e^{i\vec{k}\cdot\left(\vec{R}+\vec{x}'\right)} e^{i\vec{k}'\cdot\vec{x}'} \frac{e^{i\omega R/c}}{R} d^3x' R^2 dRd \\
&= \int e^{i\vec{k}\cdot\vec{R}} e^{i\omega R/c} RdRd \quad \int e^{i\left(\vec{k}+k'\right)\cdot\vec{x}'} d^3x' \\
&= (2\pi)^3 \int e^{i\vec{k}\cdot\vec{R}} e^{i\omega R/c} RdRd \quad \delta\left(\vec{k}+\vec{k}'\right)
\end{aligned}$$

Now put the polar axis along $\vec{k}$:

$$\begin{aligned}
\int e^{i\vec{k}\cdot\vec{R}} e^{i\omega R/c} RdRd &= \int e^{ikR\mu} e^{i\omega R/c} RdRd\mu d\phi \\
&= 2\pi \int \left. \frac{e^{ikR\mu}}{ikR} \right|_{-1}^{+1} e^{i\omega R/c} RdR \\
&= \frac{2\pi}{ik} \int_0^\infty e^{i(k+\omega/c)R} - e^{i(-k+\omega/c)R} dR \\
&= \frac{2\pi}{ik} \left. \left(\frac{e^{i(k+\omega/c)R}}{i(k+\omega/c)} - \frac{e^{i(-k+\omega/c)R}}{i(-k+\omega/c)} \right) \right|_0^\infty
\end{aligned}$$

To make the upper limit vanish, we let ω have a small, positive imaginary part, so that

$$e^{i\omega R/c} = e^{i\omega_r R/c} e^{-\omega_i R/c} \to 0 \text{ as } R \to \infty$$

Then:

$$\begin{aligned}
\int e^{i\vec{k}\cdot\vec{R}} e^{i\omega R/c} RdRd &= \frac{2\pi}{ik} \left(-\frac{1}{i(k+\omega/c)} + \frac{1}{i(-k+\omega/c)} \right) \\
&= 4\pi \frac{1}{k^2 - \omega^2/c^2}
\end{aligned}$$

Thus

$$\int\int e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'} \frac{e^{i\omega R/c}}{R} d^3x' d^3x = (2\pi)^3 4\pi \frac{1}{k^2 - \omega^2/c^2} \delta\left(\vec{k}+\vec{k}'\right)$$

and hence

$$\begin{aligned}
\int \rho(\vec{x}) \frac{\rho(\vec{x}')}{R} e^{i\omega R/c} d^3x' d^3x &= \int d^3\vec{k} \int d^3\vec{k}' \frac{e^2}{(2\pi)^6} f\left(\vec{k}\right) f\left(\vec{k}'\right) (2\pi)^3 4\pi \frac{1}{k^2 - \omega^2/c^2} \delta\left(\vec{k}+\vec{k}'\right) \\
&= \frac{e^2}{2\pi^2} \int d^3\vec{k} \frac{\left| f\left(\vec{k}\right) \right|^2}{k^2 - \omega^2/c^2}
\end{aligned}$$

and finally

$$M(\omega) = m_0 + \frac{e^2}{3c^2\pi^2} \int d^3\vec{k} \frac{\left| f\left(\vec{k}\right) \right|^2}{k^2 - \omega^2/c^2}$$

which is J equation 16.32. Letting $\omega \to 0$, we obtain the time-independent "physical" mass of the particle, including the effect of the self fields. It is

$$m = m_0 + \frac{e^2}{3c^2\pi^2} \int d^3\vec{k} \frac{\left|f\left(\vec{k}\right)\right|^2}{k^2} \tag{4}$$

For a point particle, for form factor $f\left(\vec{k}\right) = 1$ (This gives a delta-function density.). The mass function is divergent for such a particle. The effective mass

$$\begin{aligned} M(\omega) &= m + \frac{e^2}{3c^2\pi^2} \int d^3\vec{k} \left|f\left(\vec{k}\right)\right|^2 \left(\frac{1}{k^2 - \omega^2/c^2} - \frac{1}{k^2}\right) \\ &= m + \frac{e^2\omega^2}{3c^4\pi^2} \int d^3\vec{k} \left|f\left(\vec{k}\right)\right|^2 \frac{1}{k^2\left(k^2 - \omega^2/c^2\right)} \end{aligned}$$

The integral in this expression is convergent even when $f\left(\vec{k}\right) = 1$.

For a point particle:

$$\begin{aligned} M(\omega) &= m + \frac{e^2\omega^2}{3c^4\pi^2} \int d^3\vec{k} \frac{1}{k^2\left(k^2 - \omega^2/c^2\right)} \\ &= m + \frac{e^2\omega^2}{3c^4\pi^2} 4\pi \int_0^\infty \frac{dk}{\left(k^2 - \omega^2/c^2\right)} \\ &= m + \frac{2e^2}{3} \frac{\omega^2}{c^4\pi} \int_{-\infty}^\infty \frac{dk}{\left(k^2 - \omega^2/c^2\right)} \end{aligned}$$

We do the integral by contour integration. There are poles at $k = \pm\omega/c$. Recall that ω has a small, positive imaginary part, so these poles are not on the real axis. Closing in the upper half-plane, we enclose the pole at $k = +\frac{\omega}{c}$ and the result is:

$$\begin{aligned} M(\omega) &= m + \frac{2}{3} e^2 \frac{\omega^2}{c^4\pi} (2\pi i) \frac{c}{2\omega} = m + i\frac{2}{3}\frac{e^2}{c^3}\omega \\ &= m(1 + i\omega\tau) \end{aligned}$$

Looking back at the transformed equation of motion, the solution is

$$\begin{aligned} \vec{a}(\omega) &= \frac{\vec{F}_{\text{ext}}(\omega)}{M(\omega)} \\ \vec{a}(t) &= \frac{1}{\sqrt{2\pi}} \int \frac{\vec{F}_{\text{ext}}(\omega)}{M(\omega)} e^{-i\omega t} d\omega \end{aligned} \tag{5}$$

and so for a point particle, the velocity is

$$\begin{aligned} \vec{a}(t) &= \frac{1}{\sqrt{2\pi}} \int \frac{\vec{F}_{\text{ext}}(\omega)}{m(1 + i\omega\tau)} e^{-i\omega t} d\omega \\ &= \frac{1}{i\tau\sqrt{2\pi}} \int \frac{\vec{F}_{\text{ext}}(\omega)}{m(\omega - i/\tau)} e^{-i\omega t} d\omega \end{aligned}$$

We can invert the transform using the convolution theorem, where

$$f(t) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{(\omega - i/\tau)} e^{-i\omega t} d\omega$$

For $t > 0$ we close downward and $f(t) = 0$. But for $t < 0$ we must close upward, enclosing the pole at i/τ. The solution is

$$f(t < 0) = \frac{1}{\sqrt{2\pi}} 2\pi i e^{t/\tau}$$

Then

$$\begin{aligned}
\vec{a}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \vec{F}_{\text{ext}}(t-u) f(u) du \\
&= \frac{1}{im\tau\sqrt{2\pi}} \int_{-\infty}^{0} \vec{F}_{\text{ext}}(t-u) \frac{1}{\sqrt{2\pi}} 2\pi i e^{u/\tau} du \\
&= \int_{-\infty}^{0} \frac{\vec{F}_{\text{ext}}(t-u)}{m} e^{u/\tau} \frac{du}{\tau} \\
&= \int_{0}^{\infty} \frac{\vec{F}_{\text{ext}}(t+x\tau)}{m} e^{-x} dx
\end{aligned}$$

Once again we get an integral over the future.