The self-consistency problem in EM radiation

Charged particles are the sources for EM fields, but the particles respond to the fields through the Lorentz force. Accelerated particles radiate, and the energy radiated must come from the particle's energy, and so tha back-reaction of the field son the particles cannot be ignored. Let's use a simple example to see when this effect is likely to be important.

A particle with charge q and acceleration a radiates power (by the Larmor formula)

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}$$

Thus the energy lost in time t is

$$\Delta\mathcal{E} = Pt = \frac{2}{3}\frac{q^2 a^2}{c^3}t$$

If the particle starts from rest, it gains energy in time t equal to

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\,(at)^2$$

Thus the energy lost by radiation is negligible if

$$\begin{aligned} \Delta\mathcal{E} \ll \triangle \quad & K \qquad (1) \\ \frac{2}{3}\frac{q^2 a^2}{c^3}t \quad \ll \quad & \frac{1}{2}ma^2t^2 \end{aligned}$$

or

$$t \gg \frac{4}{3}\frac{q^2}{mc^3} \equiv 2\tau$$

where we have defined

$$\tau = \frac{2}{3}\frac{q^2}{mc^3} = \frac{2}{3}\frac{r_0}{c} \qquad (2)$$

where the last equality gives the result for an electron, and r_0 is the classical electron radius. Thus this effect is important for very short time-scale phenomena, or in the very first instants of a particle's acceleration. Since $\tau \propto 1/m$, the effect is greatest for the particles with smallest mass. For an electron

$$\tau\,(\text{electron}) = \frac{2}{3}\frac{\left(4.8\times10^{-10}\text{ esu}\right)^2}{\left(9\times10^{-28}\text{ gm}\right)\left(3\times10^{10}\text{ cm}\right)^3} = 6\times10^{-24}\text{ s} \qquad (3)$$

This is a very short time!

Another example. For a particle undergoing oscillatory motion with amplitude d and frequency ω_0, its energy is

$$\mathcal{E} \simeq m\omega_0^2 d^2$$

and its acceleration is

$$a \simeq \omega_0^2 d$$

With t equal to the period, equation (1) for this system takes the form

$$\begin{aligned} \frac{2}{3}\frac{q^2\left(\omega_0^2 d\right)^2}{c^3}\frac{1}{\omega_0} &\ll m\omega_0^2 d^2 \\ \omega_0 \tau &\ll 1 \end{aligned}$$

So again the effect is important for systems with very short periods.

Radiation reaction

Neglecting radiation, the equation of motion for a particle acted upon by an external force is:

$$m\frac{d\vec{v}}{dt} = \vec{F}_{\text{ext}}$$

To incorporate the energy loss due to radiation, we assume there is an effective force $\vec{F}_{\text{rad}}$ that also acts on the particle, so that the true equation of motion is:

$$m\frac{d\vec{v}}{dt} = \vec{F}_{\text{ext}} + \vec{F}_{\text{rad}}$$

The known properties of this force are:

- $\vec{F}_{\text{rad}}$ must $\to 0$ if the acceleration $a \to 0$, because then no power is radiated.
- $\vec{F}_{\text{rad}}$ should be proportional to q^2 because P is. (The sign of the charge should not appear)
- $\vec{F}_{\text{rad}}$ most likely involves the timescale τ.

Now we invoke energy conservation. The work done by this force over a time interval must equal the energy radiated. Thus

$$\int_{t_1}^{t_2} \vec{F}_{\text{rad}}.\vec{v}\ dt = -\int_{t_1}^{t_2} P\left(t\right)\ dt = -\int_{t_1}^{t_2} \frac{2}{3}\frac{q^2}{c^3}\frac{d\vec{v}}{dt}\cdot\frac{d\vec{v}}{dt}\ dt$$

Integrate by parts:

$$\int_{t_1}^{t_2} \vec{F}_{\text{rad}}.\vec{v}\ dt = -\left.\frac{2}{3}\frac{q^2}{c^3}\frac{d\vec{v}}{dt}\cdot\vec{v}\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{2}{3}\frac{q^2}{c^3}\frac{d^2\vec{v}}{dt^2}\cdot\vec{v}\ dt$$

The integrated term is zero if:

- We have uniform circular motion so that $\frac{d\vec{v}}{dt}\cdot\vec{v} = 0$ at all times
- The motion is periodic and the time interval $t_2 - t_1$ is a whole number of periods
- The acceleration lasts for a finite time less than, and included within, $t_2 - t_1$

In any of these cases, we have:

$$\int_{t_1}^{t_2} \vec{F}_{\rm rad}.\vec{v}\ dt = \int_{t_1}^{t_2} \frac{2}{3}\frac{q^2}{c^3}\frac{d^2\vec{v}}{dt^2}\cdot\vec{v}\ dt$$

or

$$\int_{t_1}^{t_2} \left(\vec{F}_{\rm rad} - \frac{2}{3}\frac{q^2}{c^3}\frac{d^2\vec{v}}{dt^2}\right).\vec{v}\ dt = 0$$

Thus in a time-averaged sense, we may say that

$$\vec{F}_{\rm rad} = \frac{2}{3}\frac{q^2}{c^3}\frac{d^2\vec{v}}{dt^2} = m\tau\frac{d^2\vec{v}}{dt^2} \tag{4}$$

This result satisfies the preoperties we previously enumerated for $\vec{F}_{\rm rad}$. The equation of motion for the particle now becomes:

$$m\left(\frac{d\vec{v}}{dt} - \tau\frac{d^2\vec{v}}{dt^2}\right) = \vec{F}_{\rm ext} \tag{5}$$

This is the Abraham-Lorentz Equation of motion.

Implications of the Abraham-Lorentz equation

To understand the implications of this, we solve the equation using an integrating factor: Let

$$\frac{d\vec{v}}{dt} = \vec{u}e^{t/\tau}$$

Then

$$\frac{d^2\vec{v}}{dt^2} = \frac{d\vec{u}}{d\tau}e^{t/\tau} + \frac{1}{\tau}\vec{u}e^{t/\tau}$$

and inserting this result into equation (5), we have

$$-m\tau\frac{d\vec{u}}{d\tau}e^{t/\tau} = \vec{F}_{\rm ext}$$

We may now integrate directly to get:

$$\vec{u} = \frac{d\vec{v}}{dt}e^{-t/\tau} = \frac{-1}{m\tau}\int \vec{F}_{\rm ext}(t')\,e^{-t'/\tau}dt'$$

To avoid an unknow integration constant, we'd like to inetgrate over a fixed interval, with the variable t at one end. But we know the effects of radiation reaction are negligible at large times, so

$$\frac{d\vec{v}}{dt} \to \frac{\vec{F}_{\rm ext}}{m} \text{ as } t \to \infty$$

Thus we use $t \to \infty$ as the other limit. Then

$$\frac{d\vec{v}}{dt} = \frac{e^{t/\tau}}{m\tau}\int_t^\infty \vec{F}_{\rm ext}(t')\,e^{-t'/\tau}dt' \tag{6}$$

An integral over the future! Letting $s = (t - t')/\tau$, this becomes

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{-1}{m}\int_0^{-\infty} \vec{F}_{\rm ext}\left(t - s\tau\right) e^s ds \\
&= \frac{1}{m}\int_{-\infty}^{0} \vec{F}_{\rm ext}\left(t - s\tau\right) e^s ds
\end{aligned}$$

To verify that this makes sense, let's expand our function

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{1}{m}\int_{-\infty}^{0}\left[\vec{F}_{\rm ext}\left(t\right) + s\tau\frac{d\vec{F}_{\rm ext}\left(t\right)}{dt} + \cdots\right] e^s ds \\
&= \frac{\vec{F}_{\rm ext}\left(t\right)}{m} e^s|_{-\infty}^{0} + \frac{\tau}{m}\frac{d\vec{F}_{\rm ext}\left(t\right)}{dt}\left(se^s - e^s\right)|_{-\infty}^{0} + \cdots \\
&= \frac{\vec{F}_{\rm ext}\left(t\right)}{m} - \frac{\tau}{m}\frac{d\vec{F}_{\rm ext}\left(t\right)}{dt}
\end{aligned}$$

The nth term involves the integral

$$\begin{aligned}
I_n &= \int_{-\infty}^{0} s^n e^s ds = ns^n e^s|_{-\infty}^{0} - n\int_{-\infty}^{0} s^{n-1} e^s ds \\
&= -nI_{n-1} = n\left(n-1\right)I_{n-2} = \left(-1\right)^n n!
\end{aligned}$$

Thus

$$\frac{d\vec{v}}{dt} = \frac{1}{m}\sum_{n=0}\left(-1\right)^n \tau^n \frac{d^n\vec{F}_{\rm ext}}{dt^n}$$

where the first term is the Newtonian result, while the remaining terms are the corrections for radiation reaction, and are increasingly negligible as n increases. We expect

$$\tau^n\frac{d^n\vec{F}_{\rm ext}}{dt^n} \sim \tau^n\frac{\vec{F}_{\rm ext}}{T^n} \ll \vec{F}_{\rm ext}$$

The integral over the future extends over a time roughly or order τ because the exponential reduces the integrand appreciable for $t' >$, and so this rather bizarre result doe not violate "macroscopic causality: The uncertainty principle restricts our knowledge:

$$\begin{aligned}
\Delta E\Delta t &\gtrsim \hbar \\
\Delta t &\gtrsim \frac{\hbar}{\Delta E}
\end{aligned}$$

Then if the change in energy is or order mc^2, the particle's rest energy, the restirtcion on observable time intervals is

$$\Delta t \gtrsim \frac{\hbar}{mc^2} = \frac{\hbar c}{e^2}\frac{e^2}{mc^3} = 137 \times \frac{3}{2}\tau \sim 200\tau$$

So the interval in the future that determines the radiation reaction effects is within the quantum uncertainty.

Now let's apply this to an electron in an atom. The electron is bound by a restoring force $\vec{F}_{\rm res} = -m\omega_0^2\vec{x}$ and so the equation of motion, including radiation reaction, is

$$\begin{aligned} m\frac{d^2}{dt^2}\vec{x} &= \vec{F}_{\rm res} + \vec{F}_{\rm rad} + \vec{F}_{\rm ext} \\ m\frac{d^2}{dt^2}\vec{x} &= -m\omega_0^2\vec{x} - m\tau\frac{d^2\vec{v}}{dt^2} + \vec{F}_{\rm ext} \end{aligned}$$

Let's begin by considering the case where $\vec{F}_{\rm ext} = 0$. The zeroth order solution, ignoring radiatin reaction, is

$$\vec{x} = \vec{x}_0 \cos\omega_0 t$$

so let's look for a solution

$$\vec{x} = \vec{x}_0 e^{-i\omega t}$$

Then stuffing in, we get

$$-m\omega^2\vec{x} + m\omega_0^2\vec{x} + m\tau\left(-i\omega\right)^3\vec{x} = 0$$

and we expect $\omega = \omega_0 + \gamma$ where $\gamma \ll \omega_0$. Thus

$$-\left(\omega_0+\gamma\right)^2 + \omega_0^2 + i\tau\left(\omega_0+\gamma\right)^3 = 0$$

and expanding to first order, we have

$$\begin{aligned} -2\omega_0\gamma + i\tau\omega_0^3 &= 0 \\ \gamma &= i\frac{(\omega_0\tau)}{2}\omega_0 = i\frac{\omega_0^2\tau}{2} \equiv i\frac{\Gamma}{2} \end{aligned}$$

Now the solution is

$$\vec{x}\left(t\right) = \vec{x}_0\cos\omega_0 t e^{-\Gamma t/2}$$

and its Fourier transform is

$$\begin{aligned} \vec{x}\left(\omega\right) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \vec{x}_0\cos\omega_0 t e^{-\Gamma t/2}e^{i\omega t}dt \\ &= \frac{1}{2\sqrt{2\pi}}\vec{x}_0\left[\frac{e^{-\Gamma t/2}e^{i(\omega+\omega_0)t}}{-\Gamma/2 + i\left(\omega+\omega_0\right)} + \frac{e^{-\Gamma t/2}e^{i(\omega-\omega_0)t}}{-\Gamma/2 + i\left(\omega-\omega_0\right)}\right]_0^\infty \\ &= \frac{1}{2\sqrt{2\pi}}\vec{x}_0\left[\frac{1}{\Gamma/2 - i\left(\omega+\omega_0\right)} + \frac{1}{\Gamma/2 - i\left(\omega-\omega_0\right)}\right] \\ &= \frac{1}{2\sqrt{2\pi}}\vec{x}_0\left[\frac{\Gamma/2 - i\left(\omega+\omega_0\right) + \Gamma/2 - i\left(\omega-\omega_0\right)}{\left(\Gamma/2\right)^2 - i\left(\omega+\omega_0\right)\Gamma/2 - i\left(\omega-\omega_0\right)\Gamma/2 - \left(\omega^2-\omega_0^2\right)}\right] \\ &= \frac{1}{\sqrt{2\pi}}\vec{x}_0\left[\frac{\Gamma/2 - i\omega}{\left(\Gamma/2\right)^2 - i\omega\Gamma - \left(\omega^2-\omega_0^2\right)}\right] \end{aligned}$$

Then we can compute the power radiated:

$$
\begin{aligned}
P(t) &= \frac{2}{3}\frac{e^2}{c^3}a(t)^2 \\
W(t) &= \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{+\infty} a(t)^2\, dt = \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{+\infty} |a(\omega)|^2\, d\omega
\end{aligned}
$$

where

$$\vec{a}(\omega) = -\omega^2 \vec{x}(\omega) = -\omega^2 \frac{1}{\sqrt{2\pi}}\vec{x}_0 \left[\frac{\Gamma/2 - i\omega}{(\Gamma/2)^2 - i\omega\Gamma - (\omega^2 - \omega_0^2)}\right]$$

and thus

$$
\begin{aligned}
\frac{dW}{d\omega} &= \frac{2}{3}\frac{e^2}{c^3}|a(\omega)|^2 \\
&= \frac{2}{3}\frac{e^2}{mc^3}\frac{m\omega^4 x_0^2}{2\pi}\frac{(\Gamma/2)^2 + \omega^2}{\omega^2\Gamma^2 + (\omega^2 - \omega_0^2)^2}
\end{aligned}
$$

Result is very peaked around $\omega = \omega_0$ so adding positive and negative frequency contributions, and noting that $\Gamma \ll \omega_0$, the power spectrum for positive (physical) frequencies is

$$
\begin{aligned}
\frac{dW}{d\omega} &= 2\frac{m\omega_0^2 x_0^2}{2\pi}\frac{\omega_0^2\tau}{\Gamma^2 + \omega_0^2(1 - \omega^2/\omega_0^2)^2} \\
&= \frac{1}{2}m\omega_0^2 x_0^2 \times \frac{2}{\pi}\frac{\Gamma}{\Gamma^2 + \omega_0^2(1 - \omega^2/\omega_0^2)^2} \\
&\simeq E_0\phi(\omega)
\end{aligned}
$$

where E_0 is the initial energy of the oscillator and

$$\phi(\omega) = \frac{2}{\pi}\frac{\Gamma}{\Gamma^2 + \omega_0^2(1 - \omega^2/\omega_0^2)^2}$$

is the line shape function.

$$\int_0^\infty \frac{2}{\pi}\frac{\Gamma}{\Gamma^2 + \omega_0^2(1 - \omega^2/\omega_0^2)^2}d\omega = \frac{2\Gamma}{\pi\omega_0^2}\int_0^\infty \frac{d\omega}{\Gamma^2/\omega_0^2 + (1 - \omega^2/\omega_0^2)^2}$$

Now let

$$
\begin{aligned}
\left(1 - \omega^2/\omega_0^2\right) &= \frac{\Gamma}{\omega_0}\tan\theta \\
-\frac{2\omega d\omega}{\omega_0^2} &= \frac{\Gamma}{\omega_0}\sec^2\theta d\theta \simeq -\frac{2d\omega}{\omega_0}
\end{aligned}
$$

As $\omega \to \infty$, $\theta \to -\pi/2$ and as $\omega \to 0$,

$$\tan\theta \to \frac{\omega_0}{\Gamma} \gg 1$$

so $\theta \sim \pi/2$. Thus

$$\begin{aligned}\int_0^\infty \phi(\omega)\, d\omega &= \frac{\Gamma}{\pi\omega_0^2}\int_{-\pi/2}^{\pi/2}\frac{\Gamma\sec^2\theta d\theta}{\Gamma^2/\omega_0^2\sec^2\theta} \\ &= \frac{1}{\pi}\pi = 1\end{aligned}$$

Thus the total energy radiated is the initial energy of the oscillator, as expected.

Now let's look at a driven oscillator. This is a model for an atom encountering an incoming electromagnetic wave.

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x + m\tau\frac{d^3x}{dt^3} - eE_0 e^{-i\omega t}$$

We expect the response to be at the driving frequency: $x = x_0 e^{-i\ t}$. Then:

$$-\omega^2 x + \omega_0^2 x - (-i\omega)^3 \tau x = -\frac{e}{m}E_0$$

and thus,

$$x(t) = \frac{e}{m}E_0\frac{1}{(\omega^2 - \omega_0^2 + i\omega^3\tau)}e^{-i\omega t}$$

where as usual the real part is assumed. Thus

$$a(t) = \frac{e}{m}E_0\frac{-\omega^2}{(\omega^2 - \omega_0^2 + i\omega^3\tau)}e^{-i\omega t}$$

and and the time-averaged radiated power is:

$$P = \frac{1}{2}\frac{2}{3}\frac{e^2}{c^3}|a(t)|^2 = \frac{1}{3}\frac{e^2}{c^3}\omega^4\frac{e^2}{m^2}E_0^2\frac{1}{(\omega^2-\omega_0^2)^2 + (\omega^3\tau)^2}$$

and the scattering cross section is:

$$\begin{aligned}\sigma &= \frac{P}{cE_0^2/8\pi} = \frac{8\pi}{3}\frac{e^4}{c^4m^2}\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega^3\tau)^2} \\ &= \sigma_T\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega^3\tau)^2} \qquad (7)\end{aligned}$$

(a) For $\omega \gg \omega_0$, we have $\sigma \approx \sigma_T$. The electron scatters radiation as if it were free.

(b)For $\omega \ll \omega_0$,

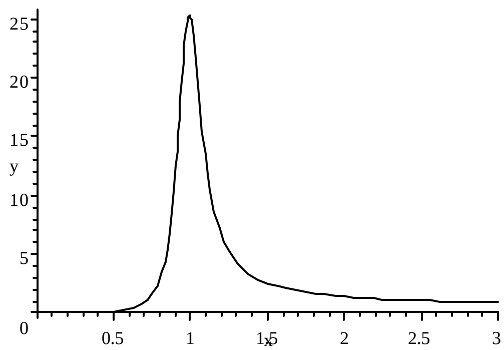
$$\sigma = \sigma_T\left(\frac{\omega}{\omega_0}\right)^4$$

and we have Rayleigh scattering.

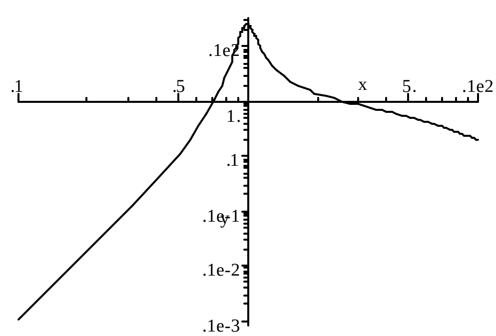
(c) For $\omega \sim \omega_0$,

$$\begin{aligned}
\sigma &= \sigma_T \frac{\omega_0^4}{\left(\omega^2 - \omega_0^2\right)^2 + \left(\omega_0^3 \tau\right)^2} = \sigma_T \frac{\omega_0^2}{\omega_0^2 \left(1 - \omega^2/\omega_0^2\right)^2 + \Gamma^2} \\
&= \frac{\sigma_T}{\tau} \frac{\pi}{2} \frac{2}{\pi} \frac{\Gamma}{\omega_0^2 \left(1 - \omega^2/\omega_0^2\right)^2 + \Gamma^2} \\
&= \frac{\sigma_T}{\tau} \frac{\pi}{2} \phi(\omega)
\end{aligned}$$

and we have an absorption line with the Lorentz profile. In the plot below, we used $\Gamma/\omega_0 = 1/5$ in equation (7).



In a log-log plot it looks like:



The total scattering cross section (frequency inegrated) for the line is:

$$\begin{aligned}
\sigma &= \frac{\sigma_T}{\tau} \frac{\pi}{2} \int_0^\infty \phi(\omega)\, d\omega = \\
&= \frac{\sigma_T}{2\tau} \pi
\end{aligned}$$

where we used our previous result for the integral over the line profile . Thus

$$\begin{aligned}\sigma &= \frac{8\pi}{3}\frac{e^4}{c^4m^2}\frac{1}{2\frac{2}{3}\frac{e^2}{mc^3}}\pi \\ &= 2\pi^2\frac{e^2}{mc} = (2\pi)\frac{\pi e^2}{mc}\end{aligned}$$

which is independent of ω_0. Thus we may write the result in terms of frequency ν as:

$$\sigma(\nu) = \frac{\pi e^2}{mc} f\varphi(\nu)$$

where

$$\int \phi(\omega)\,d\omega = \int \phi(\nu)\,2\pi d\nu = \int \varphi(\nu)\,d\omega$$

$$\begin{aligned}\varphi(\nu) &= \frac{2}{\pi}\frac{\Gamma}{\Gamma^2+\omega_0^2\left(1-\omega^2/\omega_0^2\right)^2} = 4\frac{\Gamma/2\pi}{\Gamma^2+\omega_0^2\left(1-\omega/\omega_0\right)^2\left(1+\omega/\omega_0\right)^2} \\ &\simeq 4\frac{\Gamma/2\pi}{\Gamma^2+(2\pi)^2\nu_0^2\left(1-\nu/\nu_0\right)^2(2)^2} = \frac{\Gamma/2\pi}{(2\pi)^2\left(\nu-\nu_0\right)^2+\left(\Gamma/2\right)^2}\end{aligned}$$

and

$$\int_0^\infty \phi(\nu)\,d\nu = 1$$

The fudge factor f is called the oscillator strength. It may be measured or calculated (for simple atoms) using quantum mechanics. It takes into account deviations from classical theory. Values of f are tabulated in reference works such as Allen's Astrophysical Quantities.