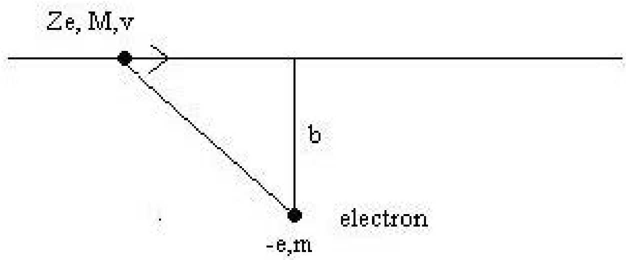
Jackson notes batch 3

1 Coulomb collisions

1.1 Basic model

Here we will consider the ineteraction of two particles through their electric and magnetic fields. We work in the rest frame of one of the particles, so we need consider only the electric field of the moving particle. We shall also consider small angle scattering. We'll clarify the meaning of "small" as we go along. These intereactions are important when fast particles interact in matter, e.g. in particle detectors.



First consider an interaction of a particle of charge Ze with an electron, as shown. We take $t = 0$ at the time of closest approach. To first order, the path of the particle remains straight, and the horizontal component of its electric field does not have any net effect on the electron. The perpendicular electric field component produced by the moving particle at the position of the electron is (notes 2 pg 6):

$$E_{\perp} = \frac{\gamma Z e b}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}$$

The reaction is impulsive (especially so when γ is large) so we make the approximation that the electron does not move during the collision. Then the impulse is:

$$\begin{aligned}
\Delta p_{\perp} &= \int_{-\infty}^{+\infty} e E_{\perp}(t)\, dt \\
&= \gamma Z e^2 b \int_{-\infty}^{+\infty} \frac{dt}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}
\end{aligned}$$

Let $\gamma vt = b\tan\theta$. Then:

$$\begin{aligned}\Delta p_\perp &= \frac{Ze^2}{bv}\int_{-\pi/2}^{+\pi/2}\frac{\sec^2\theta d\theta}{\sec^3\theta} \\ &= \frac{Ze^2}{vb}\sin\theta|_{-\pi/2}^{+\pi/2} \\ &= 2\frac{Ze^2}{vb}\end{aligned}$$

The energy transferred to the electron is

$$\mathcal{E} = \frac{p^2}{2m} = \frac{2}{m}\left(\frac{Ze^2}{vb}\right)^2 \tag{1}$$

where m is the electron mass. Here we assumed that the *electron*'s velocity is non-relativistic.

1.2 Limits of validity:

If momentum $\Delta\vec{p}$ is transferred to the electron, then the ion also has a change of momentum $-\Delta\vec{p}$. Therefore the ion is deflected through an angle:

$$\theta \simeq \frac{\Delta p}{p_{\text{ion}}} = 2\frac{Ze^2}{bv\gamma Mv} = 2\frac{Ze^2}{\gamma bMv^2} \tag{2}$$

where M is the ion mass, and this quantity must be $\ll 1$ for consistency.

During the collision the electron travels a distance less than

$$\frac{\Delta p}{2m}\Delta t$$

where Δt is the duration of the collision, approximately $b/\gamma v$. Thus we need

$$\frac{Ze^2}{bvm}\frac{b}{\gamma v} = \frac{Ze^2}{\gamma mv^2} = b_{\min} \ll b \tag{3}$$

This gives us the minimum impact parameter for which the analysis is valid. Note that if $b > b_{\text{min}}$, then the deflection of the ion will be small, as we have assumed.

1.3 Energy loss for ions passing through the material

If the electron gains energy, then the ion must lose an equal amount of energy. As the ion passes through a material, it interacts with electrons at varying impact parameters. Over a length dl of the ion's path, the number of electrons with impact parameters between b and $b + db$ is

$$dn = NZ_m 2\pi b db dl$$

where Z_m is the charge per atom of the material and N is the number density of these atoms. Then the energy lost is

$$d\mathcal{E} = \int_{b_{\min}}^{b_{\max}} \Delta\mathcal{E}\,(b)\, N Z_m 2\pi b db dl$$

and thus the energy lost per length of path is (using equation 1)

$$\begin{aligned} \frac{d\mathcal{E}}{dl} &= \int_{b_{\min}}^{b_{\max}} \frac{2}{m}\left(\frac{Ze^2}{bv}\right)^2 N Z_m 2\pi b db \\ &= \frac{4\pi}{m}\frac{Z^2 e^4}{v^2} N Z_m \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \\ &= 4\pi \frac{Z^2 e^4}{mv^2} N Z_m \ln \frac{b_{\max}}{b_{\min}} \qquad (4) \end{aligned}$$

The exact values of the maximum and minimum impact parameters are not critical, because they appear inside the slowly-varying log function. Note that the result is proportional to Z^2, so heavy ions loose energy faster.

1.4 Scattering of the ion

The incoming ion loses energy primarily by interaction with electrons in the material, but it is deflected primarily by interaction with the ions. The calculation proceeds as before with $Z_m e$ replacing e. We obtain the deflection angle for one interaction (2):

$$\theta = 2\frac{Z_m Z e^2}{\gamma b M v^2} \qquad (5)$$

A material ion interacts with incoming ions at varying impact parameters. We are interested in the number of incoming ions that emerge into a given cone of solid angle with polar angle between θ and $\theta + d\theta$. (The polar axis is along the incoming ion's direction.) This defines the *differential scattering cross section*:

$$2\pi n b db = n\frac{d\sigma}{d\Omega}d\Omega = n\frac{d\sigma}{d\Omega}2\pi \sin\theta d\theta$$

where here n is the density of incoming ions. Thus

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta}$$

But we already have an expression for $b(\theta)$ (equation 5), so

$$\begin{aligned} \left|\frac{d\sigma}{d\Omega}\right| &= \frac{b}{\sin\theta} 2\frac{Z_m Z e^2}{\gamma M v^2}\frac{1}{\theta^2} \\ &= \left(2\frac{Z_m Z e^2}{\gamma M v^2}\right)^2 \frac{1}{\theta^3 \sin\theta} \end{aligned}$$

and for small angle scattering,

$$\left|\frac{d\sigma}{d\Omega}\right| = \left(2\frac{Z_m Z e^2}{\gamma M v^2}\right)^2 \frac{1}{\theta^4}$$

This is the "well-known" result to which Jackson refers. Of course the result cannot be right as $\theta \to 0$, because we do not expect $d\sigma/d\Omega$ to be infinite. The total scattering cross section is

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_{\theta_{\min}}^{\theta_{\max}} \left(2\frac{Z_m Z e^2}{\gamma m v^2}\right)^2 \frac{1}{\theta^4} 2\pi \sin\theta d\theta$$

Since all the angles in the relevant range are small, we approximate:

$$\sigma = 2\pi \left(2\frac{Z_m Z e^2}{\gamma m v^2}\right)^2 \int_{\theta_{\min}}^{\theta_{\max}} \frac{1}{\theta^3} d\theta \simeq \left(2\frac{Z_m Z e^2}{\gamma m v^2}\right)^2 \frac{\pi}{\theta_{\min}^2} \tag{6}$$

Now we can estimate the maximum and minimum angles.

The uncertainty principle gives one limit.

$$\Delta x \Delta p \geq \hbar$$

But for the ion to interact, $\Delta x \leqslant a$, the atomic radius. Thus $\Delta p \geq \hbar/a$. Thus the actual transfer must be $\geq$ this minimum value or it is not measurable. Thus gives a limit on b :

$$\begin{aligned} 2\frac{Ze^2}{vb} &\geq \hbar/a \\ b &\leq 2\frac{Ze^2 a}{v\hbar} = b_{\max} \end{aligned}$$

and

$$\theta = \frac{\Delta p}{p} \geq \frac{\hbar}{pa} = \theta_{\min} \tag{7}$$

Thus

$$\begin{aligned} \sigma &\simeq \left(2\frac{Z_m Z e^2}{\gamma M v^2}\right)^2 \frac{\pi (pa)^2}{\hbar^2} \\ &= \pi a^2 \left(2\frac{Z_m Z e^2}{\hbar v}\right)^2 = \pi a^2 \left(2\frac{Z_m Z}{137\beta}\right)^2 \end{aligned}$$

The first part of this expression is the mechanical cross-section, and the term in parentheses is a correction which may be $\ll 1$ for fast-moving particles.

To obtain $\theta_{\max}$, note that the ion has a wavelength $\sim \hbar/p$ and this wave diffracts off the atomic nucleus. The diffraction pattern has width

$$\theta \sim \frac{\lambda}{R} = \frac{\hbar}{pR} = \theta_{\max} \tag{8}$$

where R is the radius of the nucleus. This is a reasonable estimate of $\theta_{\max}$.

1.5 Multiple collisions

Most interactions lead to small deflections, but if the ion interacts many times large deflections may be built up. The process is a random walk, with $<\theta>$ = 0, but (using result 6) the mean square deflection is

$$\begin{aligned}
<\theta^2> &= \frac{1}{\sigma}\int \theta^2 \frac{d\sigma}{d\Omega} d\Omega \\
&= \frac{1}{\sigma}\int_{\sigma_{\min}}^{\theta_{\max}} \left(2\frac{Z_m Ze^2}{\gamma Mv^2}\right)^2 \frac{1}{\theta^2} 2\pi\theta d\theta \\
&= 2\theta_{\min}^2 \ln\frac{\theta_{\max}}{\theta_{\min}} \\
&\sim 2\theta_{\min}^2 \ln\frac{a}{R} \sim 2\theta_{\min}^2 \ln\frac{\text{number}}{(AZ_m)^{1/3}}
\end{aligned}$$

where we used (7), (8), and the fact that the atomic radius is approximately $1.4a_0 Z_m^{-1/3}$ and the nuclear radius is approximately $A^{1/3}$, where A is the atomic number.

The total number of collisions per thickness t of material is

$$N = n\sigma t$$

where n is the density of target atoms. Thus the rms displacement is

$$\begin{aligned}
\theta_{rms} &= \sqrt{N}\sqrt{<\theta^2>} \\
&= \sqrt{n\sigma t}\sqrt{2\theta_{\min}^2 \ln\frac{\text{number}}{(AZ_m)^{1/3}}} \\
&= \left(2\frac{Z_m Ze^2}{\gamma mv^2}\right)\sqrt{2\pi nt}\sqrt{\ln\frac{\text{number}}{(AZ_m)^{1/3}}}
\end{aligned}$$

The distribution is a Gaussian of width θ_{rms} which attaches to the single-scattering $1/\theta^3$ distribution (see 6) at larger θs.

1.5.1 Energy loss– more accurate computation

Here we will relax the "impulsive" approximation. Instead, we view the electron as a damped harmonic oscillator:

$$\frac{d^2\vec{x}}{dt^2} + \Gamma\frac{d\vec{x}}{dt} + \omega_0^2\vec{x} = -\frac{e}{m}\vec{E}(t)$$

where the electric field is due to the incoming ion:

$$\vec{E}(t) = \frac{\gamma Zeb}{\left(b^2 + (\gamma vt)^2\right)^{3/2}}\hat{\perp} + \frac{\gamma Zevt}{\left(b^2 + (\gamma vt)^2\right)^{3/2}}\hat{\parallel} \tag{9}$$

(The restoring force is due to the nucleus and the damping is due to radiation reaction- see Chapter 16.) Taking the Fourier transform of the equation of motion, we have:

$$-\omega^2 \vec{x} + \Gamma i\omega \vec{x} + \omega_0^2 \vec{x} = -\frac{e}{m}\vec{E}(\omega)$$

so

$$\vec{x}(\omega) = -\frac{e}{m}\frac{\vec{E}(\omega)}{\omega_0^2 - \omega^2 + i\Gamma\omega} \tag{10}$$

The energy transfer is

$$\Delta\mathcal{E} = \int_{-\infty}^{+\infty} \vec{E}(\vec{x},t)\cdot\vec{j}(\vec{x},t)\,d^3x dt$$

Here $\vec{j}(\vec{x},t) = -e\vec{v}(t)\,\delta(\vec{x} - \vec{x}_e(t))$ and then

$$\begin{aligned}
\Delta\mathcal{E} &= -e\int_{-\infty}^{+\infty} \vec{E}(\vec{x}_e,t)\cdot\vec{v}(t)\,dt \\
&= -e\int_{-\infty}^{+\infty} \vec{E}(\vec{x}_e,\omega)\cdot\vec{v}^*(\omega)\,d\omega \\
&= ie\int_{-\infty}^{+\infty} \omega\vec{E}(\vec{x}_e,\omega)\cdot\vec{x}^*(\omega)\,d\omega
\end{aligned}$$

where we used Parseval's theorem. Now we stuff in the expression (10) for $\vec{x}(\omega)$.

$$\begin{aligned}
\Delta\mathcal{E} &= -i\frac{e^2}{m}\int_{-\infty}^{+\infty} \omega\vec{E}(\vec{x}_e,\omega)\cdot\frac{\vec{E}^*(\omega)}{\omega_0^2 - \omega^2 - i\Gamma\omega}d\omega \\
&= -i\frac{e^2}{m}\int_{-\infty}^{+\infty} \frac{\left|\vec{E}(\omega)\right|^2 \omega\left(\omega_0^2 - \omega^2 + i\Gamma\omega\right)}{\left(\omega_0^2 - \omega^2\right)^2 + (\Gamma\omega)^2}d\omega
\end{aligned}$$

The imaginary part integrates to zero, since the integrand is an odd function about the origin. Thus we have

$$\Delta\mathcal{E} = \frac{e^2}{m}\int_{-\infty}^{+\infty} \frac{\left|\vec{E}(\omega)\right|^2 \Gamma\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\Gamma\omega)^2}d\omega$$

If Γ is small, the integrand is sharply peaked around $\omega = \omega_0$, so we may approximate:

$$\Delta\mathcal{E} = \frac{e^2}{m}\left|\vec{E}(\omega_0)\right|^2 \int_{-\infty}^{+\infty} \frac{\Gamma\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\Gamma\omega)^2}d\omega$$

The integral may be simplified by letting $x = \omega/\Gamma$

$$I = \int_{-\infty}^{+\infty} \frac{x^2}{\left(\frac{\omega_0^2}{\Gamma^2} - x^2\right)^2 + x^2}dx$$

We may evaluate the integral by contour integration. The denominator has poles at

$$\begin{aligned} z &= \pm i\left(\frac{\omega_0^2}{\Gamma^2} - z^2\right) \\ z &= \frac{1 \pm \sqrt{1 - 4\frac{\omega_0^2}{\Gamma^2}}}{\pm 2i} = \pm\frac{i}{2}\left(1 \pm \sqrt{1 - 4\frac{\omega_0^2}{\Gamma^2}}\right) \end{aligned}$$

Since $\Gamma \ll \omega_0$,

$$z = \pm\frac{1}{2}i \pm \frac{1}{2}\sqrt{4\frac{\omega_0^2}{\Gamma^2} - 1} = \frac{1}{2}\left(\pm i \pm S\right)$$

There are 4 poles, and two of them are in the upper half plane. Call these z_1 and $z_2 = \frac{i \pm S}{2}$. Closing the contour upward, the integral along the big semicircle contributes zero, and we get:

$$\begin{aligned} I &= 2\pi i\left(\frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} + \frac{z_2^2}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)}\right) \\ &= 4\pi i\left(\frac{(i+S)^2}{2S\left[i + S - (-i+S)\right]\left[i + S - (-i - S)\right]} + \frac{(i-S)^2}{(-2S)\left(i - S - [-i+S]\right)\left(i - S - [-i - S]\right)}\right) \\ &= \frac{2\pi i}{S}\left(\frac{(i+S)^2}{(2i)\,2\,(i+S)} - \frac{(i-S)^2}{2\,(i-S)\,(2i)}\right) = \frac{\pi}{2S}\left[(i+S) - (i-S)\right] = \pi \end{aligned}$$

Thus

$$\Delta\mathcal{E} = \pi\frac{e^2}{m}\left|\vec{E}\left(\omega_0\right)\right|^2 \tag{11}$$

Finally we need the components of $\vec{E}\left(\omega\right)$. We transform (9) to get

$$\begin{aligned} E_\perp(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\gamma Z e b}{\left(b^2 + (\gamma v t)^2\right)^{3/2}}e^{i\omega t}dt \\ &= \frac{Ze}{b}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\gamma}{\left(1 + \left(\frac{\gamma v t}{b}\right)^2\right)^{3/2}}e^{i\omega t}dt \\ &= \frac{Ze}{bv}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{\left(1 + x^2\right)^{3/2}}\exp\left(i\frac{\omega b}{\gamma v}x\right)dx \\ &= \frac{Ze}{bv}\frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}\frac{\exp\left(i\frac{\omega b}{\gamma v}x\right) + \exp\left(-i\frac{\omega b}{\gamma v}x\right)}{\left(1 + x^2\right)^{3/2}}dx \\ &= \frac{Ze}{bv}\sqrt{\frac{2}{\pi}}\left[\frac{\omega b}{\gamma v}K_1\left(\frac{\omega b}{\gamma v}\right)\right] \end{aligned}$$

(GR 8.432#5). Similarly

$$\begin{aligned}
E_{\parallel}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\gamma Zevt}{\left(b^2 + (\gamma vt)^2\right)^{3/2}} e^{i\omega t} dt \\
&= \frac{Ze}{\gamma bv} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^{3/2}} \exp\left(i\frac{\omega b}{\gamma v} x\right) dx \\
&= \frac{Ze}{\gamma bv} \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{x}{(1+x^2)^{3/2}} \left[\exp\left(i\frac{\omega b}{\gamma v} x\right) - \exp\left(-i\frac{\omega b}{\gamma v} x\right)\right] dx \\
&= i\frac{Ze}{\gamma bv} \sqrt{\frac{2}{\pi}} \frac{\omega b}{\gamma v} K_0\left(\frac{\omega b}{\gamma v}\right)
\end{aligned}$$

(GR 3.754#3). Putting these results into (11), we have

$$\Delta\mathcal{E} = \frac{e^2}{m}\left(\frac{Ze}{bv}\right)^2 \left(\frac{\omega b}{\gamma v}\right)^2 \left\{\left[K_1\left(\frac{\omega b}{\gamma v}\right)\right]^2 + \frac{1}{\gamma^2}\left[K_0\left(\frac{\omega b}{\gamma v}\right)\right]^2\right\}$$

The argument of each Bessel function is

$$\xi = \frac{\omega b}{\gamma v} = \frac{b}{b_{\max}}$$

When this is small,

$$\xi K_1(\xi) \simeq 1 \text{ and } \xi K_0(\xi) \simeq \xi\left(-0.5772 - \ln(\xi/2)\right) \ll 1$$

so

$$\Delta\mathcal{E} = \frac{e^2}{m}\left(\frac{Ze}{bv}\right)^2$$

as we obtained previously (eqn. 1). But when $\xi \gg 1$, $K_n(\xi) \approx \sqrt{\pi/2\xi} e^{-\xi}$ and

$$\Delta\mathcal{E} = \frac{\pi}{2}\frac{e^2}{m}\left(\frac{Ze}{bv}\right)^2 \exp\left(-2\frac{\omega b}{\gamma v}\right)\left(1 + \frac{1}{\gamma^2}\right) \tag{12}$$

The energy transfer goes to zero exponentially as b increases.

2 Cerenkov radiation

So far we have considered only the interactions between individual particles, but in dense media the fields are affected by many atoms. We can take this into account by using the dielectric constant. We begin by writing Maxwell's equations, and taking the Fourier transform:

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f \to i\vec{k} \cdot \vec{D} = 4\pi\rho_f\left(\omega, \vec{k}\right)$$

$$\vec{\nabla}\cdot\vec{B}=0\rightarrow i\vec{k}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\rightarrow i\vec{k}\times\vec{E}=i\omega\vec{B}$$
$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{j}+\frac{1}{c}\frac{\partial\vec{D}}{\partial t}\rightarrow i\vec{k}\times\vec{B}=\frac{4\pi}{c}\vec{j}\left(\omega,\vec{k}\right)-i\frac{\omega}{c}\vec{D}$$

Now introduce the potentials:

$$\vec{B}=\vec{\nabla}\times\vec{A}\rightarrow\vec{B}\left(\omega,\vec{k}\right)=i\vec{k}\times\vec{A}\left(\omega,\vec{k}\right) \tag{13}$$

and

$$\vec{E}=-\vec{\nabla}\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\rightarrow\vec{E}\left(\omega,\vec{k}\right)=-i\vec{k}\phi+i\frac{\omega}{c}\vec{A} \tag{14}$$

Finally,

$$\vec{D}\left(\omega,\vec{k}\right)=\varepsilon\left(\omega,\vec{k}\right)\vec{E}\left(\omega,\vec{k}\right)$$

Now put the potentials into Maxwell's equations:

$$\begin{aligned} i\vec{k}\times\left(i\vec{k}\times\vec{A}\right) &= \frac{4\pi}{c}\vec{j}\left(\omega,\vec{k}\right)-\frac{i\omega}{c}\varepsilon\left(-i\vec{k}\phi+i\frac{\omega}{c}\vec{A}\right)\\ -\vec{k}\left(\vec{k}\cdot\vec{A}\right)+k^2\vec{A} &= \frac{4\pi}{c}\vec{j}\left(\omega,\vec{k}\right)-\frac{\omega}{c}\varepsilon\left(\vec{k}\phi-\frac{\omega}{c}\vec{A}\right)\end{aligned}$$

Rearranging:

$$\left(k^2-\frac{\omega^2}{c^2}\varepsilon\right)\vec{A}=\frac{4\pi}{c}\vec{j}\left(\omega,\vec{k}\right)+\vec{k}\left(\vec{k}\cdot\vec{A}-\frac{\omega}{c}\varepsilon\phi\right)$$

Now take the term in parentheses on the right hand side to be zero.

$$\vec{k}\cdot\vec{A}-\frac{\omega}{c}\varepsilon\phi=0 \tag{15}$$

This is our gauge condition. Finally we have our wave-equation for $\vec{A}$:

$$\left(k^2-\frac{\omega^2}{c^2}\varepsilon\right)\vec{A}=\frac{4\pi}{c}\vec{j}\left(\omega,\vec{k}\right) \tag{16}$$

Notice that the wave phase speed is $c/\sqrt{\varepsilon}$.

Now let's work on ϕ. Start with Gauss's law:

$$\begin{aligned} i\vec{k}\cdot\varepsilon\left(-i\vec{k}\phi+i\frac{\omega}{c}\vec{A}\right) &= 4\pi\rho_f\left(\omega,\vec{k}\right)\\ k^2\phi-\frac{\omega}{c}\vec{k}\cdot\vec{A} &= \frac{4\pi}{\varepsilon}\rho_f\left(\omega,\vec{k}\right)\end{aligned}$$

Insert the gauge condition (15) to eliminate $\vec{A}$:

$$k^2\phi-\left(\frac{\omega}{c}\right)^2\varepsilon\phi=\frac{4\pi}{\varepsilon}\rho_f\left(\omega,\vec{k}\right) \tag{17}$$

Now we have a source – the incoming ion – with

$$\begin{aligned}\rho(\vec{x},t) &= Ze\delta(\vec{x}-\vec{v}t)\\ \vec{j}(\vec{x},t) &= Zev\delta(\vec{x}-\vec{v}t)\end{aligned}$$

Transforming, we have

$$\begin{aligned}\rho\left(\vec{k},\omega\right) &= \frac{Ze}{(2\pi)^2}\int \delta(\vec{x}-\vec{v}t)\, e^{-i\left(\vec{k}.\vec{x}-\omega t\right)} d^3k d\omega\\ &= \frac{Ze}{(2\pi)^2}\int e^{-i\left(\vec{k}.\vec{v}-\omega\right)t} d\omega\\ &= \frac{Ze}{2\pi}\delta\left(\omega-\vec{k}\cdot\vec{v}\right)\end{aligned}$$

and similarly

$$\vec{j}\left(\vec{k},\omega\right) = \frac{Ze\vec{v}}{2\pi}\delta\left(\omega-\vec{k}\cdot\vec{v}\right)$$

Then the potentials are (17)

$$\phi\left(\vec{k},\omega\right) = \frac{4\pi}{\varepsilon}\frac{Ze}{2\pi}\frac{\delta\left(\omega-\vec{k}\cdot\vec{v}\right)}{k^2-\left(\frac{\omega}{c}\right)^2\varepsilon} = \frac{2}{\varepsilon}Ze\frac{\delta\left(\omega-\vec{k}\cdot\vec{v}\right)}{k^2-\left(\frac{\omega}{c}\right)^2\varepsilon}$$

and (16)

$$\vec{A} = \frac{4\pi}{c}\frac{Ze\vec{v}}{2\pi}\frac{\delta\left(\omega-\vec{k}\cdot\vec{v}\right)}{k^2-\left(\frac{\omega}{c}\right)^2\varepsilon} = \frac{\vec{v}}{c}\phi\varepsilon$$

The we can calculate the fields: From (14),

$$\begin{aligned}\vec{E}\left(\vec{k}.\omega\right) &= -i\vec{k}\phi + i\frac{\omega}{c}\vec{A}\\ &= \left(-i\vec{k}+i\frac{\omega}{c}\frac{\vec{v}}{c}\varepsilon\right)\frac{2}{\varepsilon}Ze\frac{\delta\left(\omega-\vec{k}\cdot\vec{v}\right)}{k^2-\left(\frac{\omega}{c}\right)^2\varepsilon}\end{aligned}$$

and from (13),

$$\vec{B}\left(\vec{k},\omega\right) = i\vec{k}\times\frac{\vec{v}}{c}\varepsilon\frac{2}{\varepsilon}Ze\frac{\delta\left(\omega-\vec{k}\cdot\vec{v}\right)}{k^2-\left(\frac{\omega}{c}\right)^2\varepsilon}$$

All that remains is to transform back. We want the fields at $y=b$, $x=0$. So

$$\vec{E}(\omega,\vec{x}) = \frac{-i}{(2\pi)^{3/2}}\int_{\text{all }\vec{k}\text{ space}}\left(\vec{k}-\frac{\omega}{c}\frac{\vec{v}}{c}\varepsilon\right)\frac{2}{\varepsilon}Ze\frac{\delta(\omega-k_x v)}{k^2-\left(\frac{\omega}{c}\right)^2\varepsilon}\exp(ibk_y)\ d^3k$$

We begin by doing the integration over k_x, making use of the sifting property.

$$E_y(\omega,\vec{x}) = \frac{-i}{(2\pi)^{3/2}}\frac{2}{\varepsilon}\frac{Ze}{v}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{k_y}{k_y^2+k_z^2+\frac{\omega^2}{v^2}-\left(\frac{\omega}{c}\right)^2\varepsilon}\exp(ibk_y)\;dk_y dk_z$$

Now define

$$\lambda^2 = \frac{\omega^2}{v^2}\left(1-\beta^2\varepsilon\right) \tag{18}$$

Next do the integration over k_z : Let $k_z = \sqrt{\lambda^2+k_y^2}\tan\theta$

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{dk_z}{k_y^2+k_z^2+\lambda^2} &= \int_{-\pi/2}^{\pi/2}\frac{\sqrt{\lambda^2+k_y^2}\sec^2\theta d\theta}{\left(\lambda^2+k_y^2\right)\sec^2\theta} \\ &= \frac{\pi}{\sqrt{\lambda^2+k_y^2}}\end{aligned}$$

Finally we have

$$E_y(\omega,\vec{x}) = \frac{-i}{(2\pi)^{3/2}}\frac{2}{\varepsilon}\frac{Ze}{v}\int_{-\infty}^{+\infty}\frac{\pi k_y}{\sqrt{\lambda^2+k_y^2}}\exp(ibk_y)\;dk_y$$

From GR 3.754#2

$$\int_0^{\infty}\frac{\cos(ax)}{\sqrt{\beta^2+x^2}}dx = K_0(a\beta) \qquad \mathrm{Re}(\beta)>0$$

So

$$\frac{d}{da}\int_0^{\infty}\frac{\cos(ax)}{\sqrt{\beta^2+x^2}}dx = -a\int_0^{\infty}\frac{\sin(ax)}{\sqrt{\beta^2+x^2}}dx = -\frac{d}{da}K_0(a\beta)$$

Here we have

$$\begin{aligned}\int_{-\infty}^{+\infty}dk_y\frac{k_y}{\sqrt{\lambda^2+k_y^2}}\exp(ibk_y) &= \int_0^{+\infty}dk_y\frac{k_y}{\sqrt{\lambda^2+k_y^2}}\left[\exp(ibk_y)-\exp(-ibk_y)\right] \\ &= 2i\int_0^{+\infty}dk_y\frac{k_y\sin(bk_y)}{\sqrt{\lambda^2+k_y^2}} = -2i\frac{d}{db}K_0(b\lambda) \\ &= 2i\lambda K_1(b\lambda)\end{aligned}$$

Thus

$$\begin{aligned}E_y(\omega,y=b) &= \frac{-i}{(2\pi)^{3/2}}\frac{2\pi}{\varepsilon}\frac{Ze}{v}2i\lambda K_1(b\lambda) \\ &= \sqrt{\frac{2}{\pi}}\frac{Ze}{\varepsilon}\frac{\lambda}{v}K_1(b\lambda) \\ &= \sqrt{\frac{2}{\pi}}\frac{Ze}{\varepsilon}\frac{\omega}{v^2}\sqrt{\left(1-\beta^2\varepsilon\right)}K_1(b\lambda) \qquad (19)\end{aligned}$$

The $x-$component is

$$\begin{aligned}
E_x &= \frac{-i}{(2\pi)^{3/2}} \int d^3k \left(k_x - \frac{\omega}{c}\beta\varepsilon\right) \frac{2}{\varepsilon} Ze \frac{\delta(\omega - k_x v)}{k^2 - \left(\frac{\omega}{c}\right)^2 \varepsilon} \exp(ibk_y) \\
&= \frac{-i}{(2\pi)^{3/2}} \int dk_y dk_z \left(\frac{\omega}{v} - \frac{\omega}{c}\beta\varepsilon\right) \frac{2}{\varepsilon v} Ze \frac{1}{k_y^2 + k_z^2 + \lambda^2} \exp(ibk_y) \\
&= \frac{-i\omega}{(2\pi)^{1/2}} \left(1 - \beta^2\varepsilon\right) \frac{Ze}{\varepsilon v^2} \int_{-\infty}^{+\infty} dk_y \frac{1}{\sqrt{\lambda^2 + k_y^2}} \exp(ibk_y) \\
&= -i\left(1 - \beta^2\varepsilon\right) \sqrt{\frac{2}{\pi}} \frac{Ze\omega}{\varepsilon v^2} K_0(\lambda b) \qquad (20)
\end{aligned}$$

Now we find $\vec{B}$:

$$\begin{aligned}
B_z\left(\vec{k}, \omega\right) &= -ik_y\beta 2Ze \frac{\delta\left(\omega - \vec{k}\cdot\vec{v}\right)}{k^2 - \left(\frac{\omega}{c}\right)^2 \varepsilon} = \beta\varepsilon E_y \qquad (21) \\
B_y &= ik_z\beta 2Ze \frac{\delta\left(\omega - \vec{k}\cdot\vec{v}\right)}{k^2 - \left(\frac{\omega}{c}\right)^2 \varepsilon} = -\beta\varepsilon E_z = 0 \\
B_x &= 0
\end{aligned}$$

The energy flux out of a cylinder surrounding the particle's track at radius $y = b$ is the energy lost to atoms at impact parameter $> b$. In time dt and path length dx

$$\begin{aligned}
d\mathcal{E} &= \vec{S}\cdot\vec{n} 2\pi b dx dt \\
&= -\frac{c}{4\pi} E_x B_z 2\pi b dx dt
\end{aligned}$$

Thus the energy lost per unit path length during the entire interaction is:

$$\frac{d\mathcal{E}}{dx} = \int_{-\infty}^{+\infty} -\frac{c}{2} E_x B_z b dt$$

Using Parseval's theorem, we rewrite this as an integral over the frequency:

$$\begin{aligned}
\frac{d\mathcal{E}}{dx} &= -\frac{bc}{2} \int_{-\infty}^{+\infty} E_x B_z^* d\omega \\
&= bc \int_0^{+\infty} i\left(1 - \beta^2\varepsilon\right) \sqrt{\frac{2}{\pi}} \frac{Ze\omega}{\varepsilon v^2} K_0(\lambda b)\, \beta\varepsilon \left[\sqrt{\frac{2}{\pi}} \frac{Ze}{\varepsilon} \frac{\omega}{v^2} \sqrt{\left(1 - \beta^2\varepsilon\right)} K_1(b\lambda)\right]^* d\omega \\
&= ib\frac{2}{\pi} \left(\frac{Ze}{v}\right)^2 \int_0^{+\infty} \left(\frac{1}{\varepsilon} - \beta^2\right) \omega\lambda^* K_0(\lambda b) \left[K_1(b\lambda)\right]^* d\omega
\end{aligned}$$

where we have taken ε to be real. We also reduced the integral to the positive range of ω because the integrand is even in ω. (The result in terms of Bessel functions requires $\mathrm{Re}\lambda > 0$. See additional notes that get the result for λ imaginary.)

The density effects are important only when $\beta^2\varepsilon$ approaches 1.

Let's look at the energy deposited at large distances from the particle track ($\lambda b \gg 1$). Then we may use the large argument approximation to each Bessel function.

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z}$$

Thus

$$\begin{aligned}\frac{d\mathcal{E}}{dx} &= ib\frac{2}{\pi}\left(\frac{Ze}{v}\right)^2 \int_0^\infty \left(\frac{1}{\varepsilon} - \beta^2\right) \frac{\pi\omega\lambda^*}{2b\sqrt{\lambda\lambda^*}} e^{-b(\lambda+\lambda^*)} d\omega \\ &= i\left(\frac{Ze}{c}\right)^2 \int_0^\infty \omega\left(\frac{1}{\beta^2\varepsilon} - 1\right)\sqrt{\frac{\lambda^*}{\lambda}} e^{-b(\lambda+\lambda^*)} d\omega\end{aligned}$$

The result decreases exponentially with b if $\lambda + \lambda^*$ has a real part. But if λ is pure imaginary, then $\lambda + \lambda^* = 0$, and then

$$\frac{d\mathcal{E}}{dx} = \left(\frac{Ze}{c}\right)^2 \int \omega\left(1 - \frac{1}{\beta^2\varepsilon}\right) d\omega \tag{22}$$

The result is real and independent of b! Energy escapes to infinity as radiation.

The condition λ =imaginary reduces to

$$\beta^2 > \frac{1}{\varepsilon}$$

or equivalently:

$$v > v_\phi = \frac{c}{\sqrt{\varepsilon}}$$

the phase speed for light in the material. The range of integration is now restricted to those values of ω such that $\beta^2 > 1/\varepsilon(\omega)$

Finally we look at the fields in this limit. From (20), (19) and (21),

$$\begin{aligned}E_y &= \sqrt{\frac{2}{\pi}}\frac{Ze}{\varepsilon}\frac{\omega}{v^2}\sqrt{(1-\beta^2\varepsilon)}\sqrt{\frac{\pi}{2b\lambda}}e^{-b\lambda} \\ E_x &= -i\left(1-\beta^2\varepsilon\right)\sqrt{\frac{2}{\pi}}\frac{Ze\omega}{\varepsilon v^2}\sqrt{\frac{\pi}{2b\lambda}}e^{-b\lambda} \\ \vec{B} &= \varepsilon\vec{\beta}\times\vec{E}\end{aligned}$$

The fields have the correct magnitude and direction for radiation fields. The wave propagates at an angle θ_c to the direction of motion of the particle, where

$$\begin{aligned}\tan\theta_c &= \frac{-\left|E_\parallel\right|}{|E_\perp|} = \frac{-\left|\left(1-\beta^2\varepsilon\right)\sqrt{\frac{2}{\pi}}\frac{Ze\omega}{\varepsilon v^2}\sqrt{\frac{\pi}{2b\lambda}}e^{-b\lambda}\right|}{\left|\sqrt{\frac{2}{\pi}}\frac{Ze}{\varepsilon}\frac{\omega}{v^2}\sqrt{\left(1-\beta^2\varepsilon\right)}\sqrt{\frac{\pi}{2b\lambda}}e^{-b\lambda}\right|} \\ &= -\left|\sqrt{1-\beta^2\varepsilon}\right|\end{aligned}$$

Remember that $\beta^2\varepsilon > 1$. Then

$$\sec^2\theta_c = 1 + \tan^2\theta_c = 1 + \left(\beta^2\varepsilon - 1\right) = \beta^2\varepsilon$$

and so

$$\cos\theta_c = \frac{1}{\beta\sqrt{\varepsilon}} \tag{23}$$

This is the Cerenkov angle.