Physics 705- Notes batch 2

1 Relativistic E&M

Now we want to discover the set of tensors and the covariant relations that govern E&M.

We begin with the current 4-vector which is:

$$\mathbf{J} = \left(c\rho, \vec{j}\right) \tag{1}$$

Note that each component is dimensionally equivalent. The charge conservation law is

$$\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0 = \frac{\partial c\rho}{\partial ct} + \vec{\nabla} \cdot \vec{j} \tag{2}$$

where the term on the left is the 4-divergence of the 4-vector **J.**

Next we conjecture the the 4-potential is given by

$$\mathbf{A} = \left(\phi, \vec{A}\right) \tag{3}$$

We find that the Lorentz gauge condition is obtained by setting the 4-divergence to zero:

$$\frac{\partial A^{\mu}}{\partial x^{\mu}} = 0 = \frac{\partial \phi}{\partial ct} + \vec{\nabla} \cdot \vec{A} \tag{4}$$

We obtain the wave equation from the wave operator

$$\Box^2 = \partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

Then

$$\begin{aligned} \Box^2 \mathbf{A} &= \frac{4\pi}{c}\mathbf{J} \\ \partial^{\mu}\partial_{\mu} A^{\alpha} &= \frac{4\pi}{c} J^{\alpha} \end{aligned} \tag{5}$$

This gives us both of the relations we've seen before:

$$\begin{aligned} \alpha &= 0: \qquad \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 4\pi\rho \\ \alpha &= i: \qquad \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{j} \end{aligned}$$

The potential transforms with the Lorentz transformation according to the usual rule for vectors. Note particularly that the charge density is NOT a scalar!

Now for the fields. First note the relations between the fields and the potentials:

$$\vec{B} = \vec{\nabla} \times \vec{A}; \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$

We know that curls are usually associated with antisymmetric tensors, so we define the field tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{6}$$

This tensor is antisymmetric, so it has zeros along the diagonal. Recall that the gradient form has components

$$\partial_\alpha = \left(\frac{\partial}{\partial ct}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

so the corresponding vector has components

$$\partial^\alpha = g^{\alpha\beta}\partial_\beta = \left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right)$$

Thus

$$\begin{aligned} F^{10} &= \partial^1 A^0 - \partial^0 A^1 = -F^{01} \\ &= -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial ct} = E_x \end{aligned}$$

Similarly we find $F^{20} = E_y$ and $F^{30} = E_z$.

Now look at

$$\begin{aligned} F^{21} &= \partial^2 A^1 - \partial^1 A^2 = -F^{12} \\ &= -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} = B_z \end{aligned}$$

Thus we have the components of the tensor:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{7}$$

Now we can find the transformed field components by transforming the tensor in the usual way:

$$\overline{F}^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$$

or, in matrix notation:

$$
\begin{aligned}
\overline{F} &= \Lambda F \Lambda^T \\
&= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x\gamma\beta & -E_x\gamma & -E_y & -E_z \\ E_x\gamma & -E_x\gamma\beta & -B_z & B_y \\ E_y\gamma - B_z\gamma\beta & -E_y\gamma\beta + B_z\gamma & 0 & -B_x \\ E_z\gamma + B_y\gamma\beta & -E_z\gamma\beta - B_y\gamma & B_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\gamma^2 E_x + \gamma^2\beta^2 E_x & -E_y\gamma + B_z\gamma\beta & -E_z\gamma - B_y\gamma\beta \\ -\gamma^2\beta^2 E_x + \gamma^2 E_x & 0 & E_y\gamma\beta - B_z\gamma & E_z\gamma\beta + B_y\gamma \\ E_y\gamma - B_z\gamma\beta & -E_y\gamma\beta + B_z\gamma & 0 & -B_x \\ E_z\gamma + B_y\gamma\beta & -E_z\gamma\beta - B_y\gamma & B_x & 0 \end{pmatrix}
\end{aligned}
$$

But $\gamma^2\left(1-\beta^2\right) = 1$, so

$$
\overline{F} = \begin{pmatrix} 0 & -E_x & -\gamma\left(E_y - \beta B_z\right) & -\gamma\left(E_z + \beta B_y\right) \\ E_x & 0 & -\gamma\left(B_z - E_y\beta\right) & \gamma\left(B_y + E_z\beta\right) \\ \gamma\left(E_y - \beta B_z\right) & \gamma\left(B_z - E_y\beta\right) & 0 & -B_x \\ \gamma\left(E_z + \beta B_y\right) & -\gamma\left(B_y + E_z\beta\right) & B_x & 0 \end{pmatrix}
$$

Thus we have the transformation rules:

> The components of $\vec{E}$ and $\vec{B}$ parallel to the relative velocity are unchanged, and the perpendicular components transform as:

$$\overline{\vec{E}}_\perp = \gamma\left(\vec{E}_\perp + \vec{\beta}\times\vec{B}\right) \tag{8}$$

and

$$\overline{\vec{B}}_\perp = \gamma\left(\vec{B}_\perp - \vec{\beta}\times\vec{E}\right) \tag{9}$$

We'll come back to these results in a moment.

Next we want to write Maxwell's equations in covariant form. We can group the equations into the source-free equations

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{B} &= 0 \\
\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} &= 0
\end{aligned}
$$

and the equations with sources

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 4\pi\rho \\
\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &= \frac{4\pi\vec{j}}{c}
\end{aligned}
$$

The second pair are obtained from the covariant relation

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{10}$$

For example, with $\beta = 0$ we have

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi}{c} c\rho$$

and with $\beta = 1$

$$\begin{aligned}
\partial_\alpha F^{\alpha 1} &= \frac{4\pi}{c} J^1 \\
\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} &= \frac{4\pi}{c} J^1 \\
\frac{\partial}{\partial ct}(-E_x) + \frac{\partial}{\partial y}(B_z) + \frac{\partial}{\partial z}(-B_y) &= \frac{4\pi}{c} J_x
\end{aligned}$$

This equation is the $x-$component of

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

The other components follow similarly.

Two obtain the source-free equations we first define the dual tensor

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \tag{11}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is defined similarly to ϵ_{ijk}. ($\varepsilon^{\alpha\beta\gamma\delta} = 1$ if $\alpha\beta\gamma\delta =$ an even permutation of 0123, and so on.) First compute the covariant components of the field tensor:

$$\begin{aligned}
F_{\alpha\beta} &= g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}
\end{aligned} \tag{12}$$

and then we have, for example, the dual tensor components:

$$\mathcal{F}^{01} = \frac{1}{2}\varepsilon^{01\gamma\delta} F_{\gamma\delta} = \frac{1}{2}(F_{23} - F_{32}) = F_{23} = -B_x$$

and

$$\mathcal{F}^{12} = \frac{1}{2}\varepsilon^{12\gamma\delta}F_{\gamma\delta} = \frac{1}{2}\varepsilon^{1203}F_{03} + \frac{1}{2}\varepsilon^{1230}F_{30}$$

To get 1203 from 0123 We have to do two interchanges: First interchange 2 and 0 to get 1023 then interchange 1 and 0 to get 0123. Thus this is an even permutation and $\varepsilon^{1203} = +1$. Thus

$$\mathcal{F}^{12} = \frac{1}{2}\left(F_{03} - F_{30}\right) = E_z$$

Thus

$$\mathcal{F} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{13}$$

Then the remaining two Maxwell equations may be written in covariant form as

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \tag{14}$$

Finally we note the Lorentz force may be written in covariant form as

$$F^\alpha = \frac{q}{c}F^{\alpha\beta}u_\beta$$

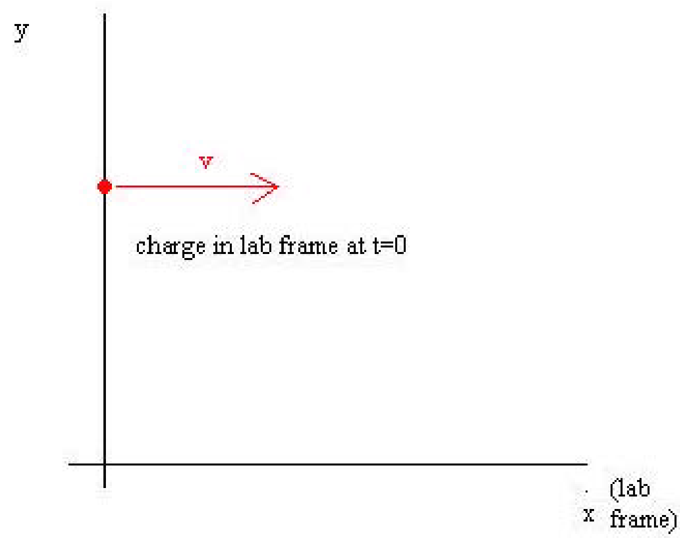
For example

$$\begin{aligned} F_x &= \frac{q}{c}\left(F^{1\beta}u_\beta\right) = \frac{q}{c}\gamma\left[E_x c + \left(-B_z\right)\left(-u_y\right) + B_y\left(-u_z\right)\right] \\ &= q\gamma\left[E_x + \left(\frac{\vec{u}}{c}\times\vec{B}\right)_x\right] \end{aligned}$$

This is the non-relativistic result in the limit $\beta \to 0$, $\gamma \to 1$.

2 Fields due to a moving point charge

Let a charge q move at constant velocity $\vec{v}$ in the lab frame, and let us set up coordinates as shown in the diagram.



In its own rest frame, the field due to a point charge obeys the usual Coulomb law:

$$\vec{E} = \frac{q}{r^2}\hat{r}, \qquad \vec{B} = 0.$$

Thus the field at the lab origin in the charge's frame is

$$\overline{E}_x = -\frac{qv\bar{t}}{\left(\overline{y}^2 + \left(v\bar{t}\right)^2\right)^{3/2}}, \quad \overline{E}_y = -\frac{q\overline{y}}{\left(\overline{y}^2 + \left(v\bar{t}\right)^2\right)^{3/2}}, \quad \overline{E}_z = 0$$

Now we transform to the lab frame using equations (8) and (9) with velocity in the negative-$x-$direction:

$$\begin{aligned} E_x &= \overline{E}_x = . -\frac{qv\bar{t}}{\left(\overline{y}^2 + \left(v\bar{t}\right)^2\right)^{3/2}} \\ E_y &= \gamma\overline{E}_y = -\frac{\gamma q\overline{y}}{\left(\overline{y}^2 + \left(v\bar{t}\right)^2\right)^{3/2}} \\ E_z &= 0 \end{aligned}$$

and

$$
\begin{aligned}
B_x &= \gamma\left(-\vec{\beta}\times\vec{\overline{E}}\right)_x = 0 \\
B_y &= \gamma\left(-\vec{\beta}\times\vec{\overline{E}}\right)_y = -\gamma\beta\overline{E}_z = 0 \\
B_z &= \gamma\left(-\vec{\beta}\times\vec{\overline{E}}\right)_z = \gamma\beta\overline{E}_y = -\frac{\gamma\beta q\overline{y}}{\left(\overline{y}^2+\left(v\overline{t}\right)^2\right)^{3/2}}
\end{aligned}
$$

Now we still need to transform the coordinates to the lab frame:

$$\overline{x} = \gamma\left(x - vt\right); \quad \overline{y} = y \text{ and } c\overline{t} = \gamma\left(ct - \beta x\right)$$

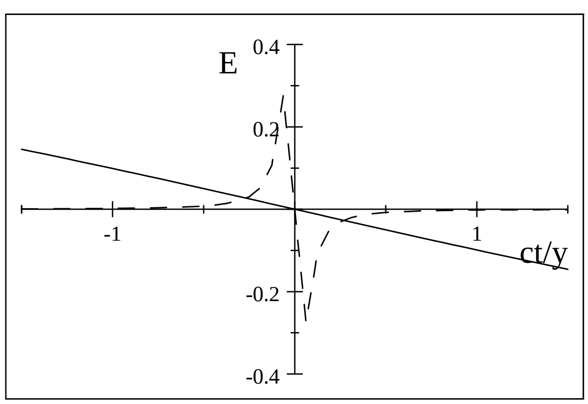
Since our observation point is at the lab origin, $x = 0$. Thus

$$
\begin{aligned}
E_x &= -\frac{q\beta\gamma ct}{\left(y^2+\left(\beta\gamma ct\right)^2\right)^{3/2}} \\
E_y &= -\frac{\gamma q\overline{y}}{\left(y^2+\left(\beta\gamma ct\right)^2\right)^{3/2}} \\
B_z &= -\frac{\gamma\beta q\overline{y}}{\left(y^2+\left(\beta\gamma ct\right)^2\right)^{3/2}} = \beta E_y
\end{aligned}
$$

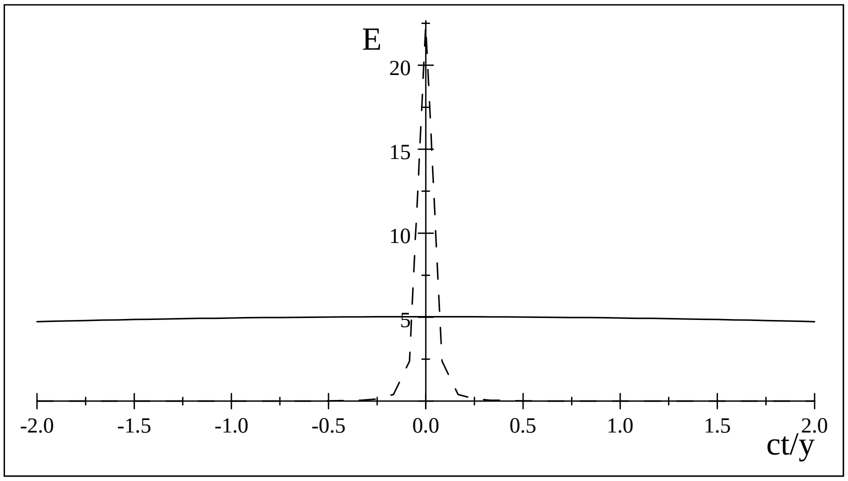
First check that these results are correct in the non-relativistic limit ($\beta \to 0$, $\gamma \to 1$). The interesting result is the relativistic limit as γ becomes very large. The fields become impulsive, (large for a very small time interval). The fields fall rapidly to zero for

$$|t| > \frac{y}{\beta\gamma c}$$

which is very small when γ is large. In addition, the magnitude of the fields becomes large for $t \approx 0$. The diagram shows the field components as functions of time for $\beta = 0.1$ ($\gamma = 1.005$)and $\beta = 0.999$ ($\gamma = 22.366$).



$E_x y^2/q$ versus ct/y. Solid: $\beta = 0.1$, Dashed, $\beta = 0.999$



$E_y y^2/q$ versus ct/y. Solid: $\beta = 0.1$ multiplied by 5, Dashed, $\beta = 0.999$

3 Invariants

We can form several invariants from the field tensors $F^{\alpha\beta}$ and $\mathcal{F}^{\alpha\beta.}$ First we evaluate

$$F^{\alpha\beta}F_{\alpha\beta} = B^2 - E^2$$

Thus if we have a pure electric field in one frame, $E > B$ in all other frames. The fields maintain their dominantly electric character, and similarly for magnetic fields.

A second invariant is

$$F_{\alpha\beta}\mathcal{F}^{\alpha\beta} = -\vec{E} \cdot \vec{B}$$

Since this product is invariant, if either of $\vec{E}$ or $\vec{B}$ is zero in one frame, the fields in any other frame are perpendicular.

Check that these results hold true for the point charge fields we calculated above.

A special case is the EM wave, for which both invariants are zero: ($\vec{E} \perp \vec{B}$ and $E = B$). These properties are therefore true in all frames.