Jackson Notes part 2 January 2020

1 Energy in the Electromagnetic field

1.1 Electrostatic case

We put a charge q in a region where there is an electrostatic field $\vec{E}$. Then the force acting on the charge is $\vec{F} = q\vec{E}$ and the work done *by* the fields *on* the charge as it moves from A to B along path C is

$$\begin{aligned} W &= q\int_A^B \vec{E}\cdot d\vec{\ell} \\ &= -q\int_A^B \vec{\nabla}\Phi\cdot d\vec{\ell} \\ &= -q\int_A^B d\Phi = q\left(\Phi_A - \Phi_B\right) \end{aligned}$$

independent of the path C. Since the work done by the fields must decrease the stored energy, we may interpret the term $q\Phi$ as the potential energy of the system comprising the charge q and the fields $\vec{E}$.

$$W = U_i - U_f = q\Phi_A - q\Phi_B$$

For a system of point charges q_i at positions $\vec{x}_i$, we may write the potential as

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\left|\vec{x}-\vec{x}_i\right|}$$

When we compute the potential at the position of a charge q_j we do not include the potential due to charge q_j (charges do not exert forces on themselves, or equivalently, we ignore the infinite self-energy of the charge). Thus the energy of charge q_j in the presence of all the other charges is

$$U_j = q_j\Phi\left(\vec{x}_j\right) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1,\ i\neq j}^{N}\frac{q_i q_j}{\left|\vec{x}-\vec{x}_i\right|} = \sum_{i=1,\ i\neq j}^{N} U_{ij} \tag{1}$$

Thus we see that the energy U_j is the sum of the energies U_{ij} of all the pairs of charges we can form with charge j. The energy of the total distribution appears to be

$$U = \sum_{j=1}^{N} U_j = \sum_{j=1}^{N}\sum_{i=1,\ i\neq j}^{N} U_{ij}$$

However, we have to be careful not to double count pairs of charges. ($U_{ij} = U_{ji}$, but we only want one of them.) (See Lea and Burke Ch 25 for more on this).

So the correct expression is

$$U=\frac{1}{4\pi\varepsilon_0}\sum_{j>i}\sum_{i=1,\ i\neq j}^{N}\frac{q_iq_j}{|\vec{x}-\vec{x}_i|}=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\sum_{i=1,\ i\neq j}^{N}\frac{q_iq_j}{|\vec{x}-\vec{x}_i|}\tag{2}$$

Next we look at what happens when the charge distribution is continuous. We replace q_i with the charge element $dq=\rho(\vec{x})\,d^3\vec{x}$ and q_j with $\rho(\vec{x}')\,d^3\vec{x}'$ and replace each sum with an integral to obtain

$$U=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\int\int\frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}d^3\vec{x}'=\frac{1}{2}\int_{\text{all space}}\rho(\vec{x})\,\Phi(\vec{x})\,d^3\vec{x}\tag{3}$$

where we used eqn (29) in Notes 1 for $\Phi(\vec{x})$. Notice, though, that we have no way to express the constraint $i\neq j$, and the self-energy is now *included.* Using Poisson's equation (Notes 1 eqn 32), we may eliminate the charge density and obtain the energy in terms of the fields alone:

$$\begin{aligned}U &= \frac{1}{2}\int\left(-\varepsilon_0\nabla^2\Phi\right)\Phi(\vec{x})\ d^3\vec{x}\\ &= -\frac{\varepsilon_0}{2}\left[\int\vec{\nabla}\cdot\left(\Phi\vec{\nabla}\Phi\right)\ dV-\int\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi\ dV\right]\\ &= -\frac{\varepsilon_0}{2}\left[\int_{S_\infty}\Phi\vec{\nabla}\Phi\cdot\hat{n}\ dA-\int\vec{E}\cdot\vec{E}\ dV\right]\end{aligned}$$

If the sources of the fields are localized, then the potential decreases at least as fast as $1/r$ as $r\to\infty$, and so the surface integral is zero. (It goes as $\frac{1}{r}\frac{1}{r^2}r^2=\frac{1}{r}$ as $r\to\infty$.) Thus

$$U=\frac{\varepsilon_0}{2}\int_{\text{all space}}E^2\ dV$$

and the energy density is

$$u_E=\frac{\varepsilon_0}{2}E^2\tag{4}$$

Extension to fields in media (see Jackson §4.7– we'll do more with this later). We replace $\varepsilon_0\vec{E}$ with $\vec{D}$

$$u_E=\frac{1}{2}\vec{E}\cdot\vec{D}\tag{5}$$

and this result is true in media with $\varepsilon\neq\varepsilon_0$ as well.

{Digression that shows the equivalence of (5) and (3).

$$\begin{aligned}U &= \frac{1}{2}\int\vec{E}\cdot\vec{D}\ dV=-\frac{1}{2}\int\vec{\nabla}\Phi\cdot\vec{D}\ dV=-\frac{1}{2}\int\left[\vec{\nabla}\cdot\left(\Phi\vec{D}\right)-\Phi\vec{\nabla}\cdot\vec{D}\right]\ dV\\ &= -\frac{1}{2}\left[\oint_{S_\infty}\Phi\vec{D}\cdot\hat{n}dA-\int\Phi\rho\ dV\right]=\frac{1}{2}\int\rho(\vec{x})\,\Phi(\vec{x})\ dV\end{aligned}$$

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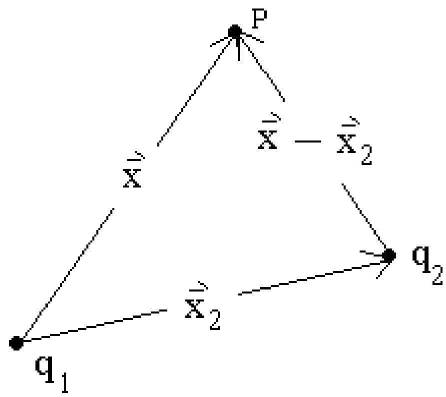
Example: energy of a pair of point charges.

Two charges, q_1 and q_2, are separated by the displacement $\vec{x}_2$. To simplify the calculation we can put the origin at the position of one of the charges, q_1.



Then the electric field at point P is

$$\vec{E}(\vec{x}) = \vec{E}_1 + \vec{E}_2 = kq_1 \frac{\vec{x}}{|\vec{x}|^3} + kq_2 \frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|^3}$$

and the energy density (4) is

$$\begin{aligned} u(\vec{x}) &= \frac{k^2 \varepsilon_0}{2} \left(q_1 \frac{\vec{x}}{|\vec{x}|^3} + q_2 \frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|^3} \right) \cdot \left(q_1 \frac{\vec{x}}{|\vec{x}|^3} + q_2 \frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|^3} \right) \\ &= k^2 \frac{\varepsilon_0}{2} \left[\frac{q_1^2}{|\vec{x}|^4} + \frac{q_2^2}{|\vec{x} - \vec{x}_2|^4} + 2q_1 q_2 \frac{\vec{x}}{|\vec{x}|^3} \cdot \frac{(\vec{x} - \vec{x}_2)}{|\vec{x} - \vec{x}_2|^3} \right] \end{aligned}$$

The first two terms are the self-energy terms, and the volume integral of each is infinite if the charges are truly points.[1] The third term is the *interaction energy.* It is the only term that changes as the charges move, and thus (classically) it is the only energy we can extract from the system.

$$\begin{aligned} U_{\text{int}} &= k^2 \frac{\varepsilon_0}{2} \int 2q_1 q_2 \frac{\vec{x}}{|\vec{x}|^3} \cdot \frac{(\vec{x} - \vec{x}_2)}{|\vec{x} - \vec{x}_2|^3} d^3\vec{x} \\ &= k^2 \varepsilon_0 q_1 q_2 \int \vec{\nabla} \frac{1}{|\vec{x}|} \cdot \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}_2|} d^3\vec{x} \\ &= k^2 \varepsilon_0 q_1 q_2 \int \left[\vec{\nabla} \cdot \left(\frac{1}{|\vec{x} - \vec{x}_2|} \vec{\nabla} \frac{1}{|\vec{x}|} \right) - \frac{1}{|\vec{x} - \vec{x}_2|} \vec{\nabla} \cdot \vec{\nabla} \frac{1}{|\vec{x}|} \right] d^3\vec{x} \\ &= k^2 \varepsilon_0 q_1 q_2 \int_{S_\infty} \frac{1}{|\vec{x} - \vec{x}_2|} \vec{\nabla} \frac{1}{|\vec{x}|} \cdot \hat{n}\, dA - \int \frac{1}{|\vec{x} - \vec{x}_2|} \nabla^2 \frac{1}{|\vec{x}|} d^3\vec{x} \end{aligned}$$

[1] Note that the infinity arises at $r = 0 :$ $\int_0^\infty \frac{1}{r^4} 4\pi r^2 dr = -\frac{4\pi}{r}\Big|_0^\infty$

The surface integral is zero (integrand goes as $1/r^3$ while dA goes as r^2 as $r \to \infty$), so, using Lea eqn 6.26, we have

$$\begin{aligned} U_{\text{int}} &= k^2 4\pi\varepsilon_0 q_1 q_2 \int \frac{1}{|\vec{x}-\vec{x}_2|}\delta(\vec{x})\, d^3\vec{x} \\ &= k\frac{q_1 q_2}{|\vec{x}_2|} \end{aligned}$$

This is the expected result. Compare with eqn (1).

1.2 Magnetic energy

The work done by the fields on moving charges (currents), per unit time, is

$$P = \frac{\delta W}{\delta t} = \int \vec{j}\cdot\vec{E}\; dV$$

When the currents are confined to wires, the expression simplifies

$$P = \frac{\delta W}{\delta t} = \sum_{\text{loops}} \int_{C_i} I\vec{E}\cdot d\vec{\ell}$$

We may use Stokes theorem to re-express the line integral:

$$\frac{\delta W}{\delta t} = \sum_{\text{loops}} I \int_{S_i} \left(\vec{\nabla}\times\vec{E}\right)\cdot\hat{n}\; dA$$

where S_i is a surface spanning loop i, and then from Faraday's law

$$\frac{\delta W}{\delta t} = \sum_{\text{loops}} I \int_{S_i} \left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot\hat{n}\; dA$$

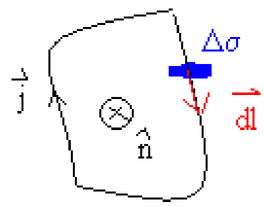
So the work done *by* the fields is

$$\begin{aligned} \delta W &= -\sum_{\text{loops}} I \int_{S_i} \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\; dA\delta t \\ &= -\sum_{\text{loops}} I\; \delta\Phi_{B,i} \end{aligned}$$

where $\Phi_{B,i}$ is the magnetic flux through surface S_i. Work done by the fields decreases the stored energy, $\delta W = -\delta U$. Thus

$$\delta U = \sum_{\text{loops}} I\; \delta\Phi_{B,i}$$

Now we let the charge density build up infinitely slowly $\left(\frac{\partial\rho}{\partial t} \simeq 0 \text{ and thus } \vec{\nabla}\cdot\vec{j} \simeq 0\right)$. This means the "field lines" of $\vec{j}$ are closed loops to which we can apply the result we have just derived. The current $I = j\Delta\sigma$ where $\Delta\sigma$ is a small area element perpendicular to $\vec{j}$. Thus



$$
\begin{aligned}
\delta U &= \sum_{\text{loops}} j\Delta\sigma \int \delta\vec{B}\cdot\hat{n}\; dS \\
&= \sum_{\text{loops}} j\Delta\sigma \int \left(\vec{\nabla}\times\delta\vec{A}\right)\cdot\hat{n}\; dS \\
&= \sum_{\text{loops}} j\Delta\sigma \int_{\text{loop } i} \delta\vec{A}\cdot d\vec{\ell} = \sum_{\text{loops}} \int_{\text{loop } i} \vec{j}\cdot\delta\vec{A}\; d\ell\Delta\sigma \\
&= \int \vec{j}\cdot\delta\vec{A}\; dV \qquad (6)
\end{aligned}
$$

where we used the fact that $\vec{j}$ is parallel to $d\vec{\ell}$. Next we use Ampere's law to write $\vec{j}$ in terms of the fields. Remember: we are changing things infinitely slowly so $\frac{\partial\vec{E}}{\partial t} \simeq 0$.

$$\delta U = \int \left(\vec{\nabla}\times\vec{H}\right)\cdot\delta\vec{A}\; dV$$

Now, from the cover,

$$\vec{\nabla}\cdot\left(\vec{b}\times\vec{a}\right) = \vec{a}\cdot\left(\vec{\nabla}\times\vec{b}\right) - \vec{b}\cdot\left(\vec{\nabla}\times\vec{a}\right)$$

Thus, with $\vec{a} = \delta\vec{A}$ and $\vec{b} = \vec{H}$, we have

$$\delta U = \int \left[\vec{\nabla}\cdot\left(\vec{H}\times\delta\vec{A}\right) + \vec{H}\cdot\left(\vec{\nabla}\times\delta\vec{A}\right)\right]\; dV$$

We use the usual trick of converting the first integral to a surface integral, and arguing that it is zero. (Note that $\vec{A}\to 0$ at least as fast as $1/r^2$ for localized $\vec{j}$, while H goes as $1/r^3$.) Then

$$\delta U = \int \vec{H}\cdot\delta\vec{B}\; dV$$

For a linear medium, $\vec{H}\propto\vec{B}$, and so

$$\delta U = \frac{1}{2}\delta\int \vec{H}\cdot\vec{B}\; dV$$

and thus

$$U = \frac{1}{2}\int \vec{H}\cdot\vec{B}\ dV$$

and the energy density is

$$u_B = \frac{1}{2}\vec{H}\cdot\vec{B}$$

In vacuum

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{\mu_0}{2}H^2$$

Combining with the electric energy, we get

$$u_{\text{em}} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \tag{7}$$

If $\bar{J}$ and $\vec{A}$ are also related linearly, as in notes 1 eqn (21), then we also have, from (6),

$$U = \frac{1}{2}\int \vec{j}\cdot\vec{A}\ dV$$

1.3 Conservation of energy, general case

When there exists a current distribution $\vec{j}$ and field $\vec{E}$, the fields do work per unit volume to maintain the current at a rate

$$P = \vec{j}\cdot\vec{E}$$

Using Ampere's law, we may eliminate $\vec{j}$. This time we allow arbitrary time variation.

$$\int_V \vec{j}\cdot\vec{E}\ dV = \int_V \left(\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{E}\ dV$$

Substitute for $\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right)$ using

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) = \vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{H}\right)$$

and use Faraday's law and the divergence theorem to get:

$$\begin{aligned}\int_V \vec{j}\cdot\vec{E}\ dV &= \int_V \left[\vec{H}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) - \frac{\partial\vec{D}}{\partial t}\cdot\vec{E}\right]\ dV \\ &= \int_V \left[\vec{H}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \frac{\partial\vec{D}}{\partial t}\cdot\vec{E}\right]\ dV - \oint_S \left(\vec{E}\times\vec{H}\right)\cdot\hat{n}\ dA\end{aligned}$$

In a linear medium, we can simplify the term in square brackets:

$$\int_V \vec{j}\cdot\vec{E}\ dV = -\frac{1}{2}\int_V \frac{\partial}{\partial t}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)\ dV - \oint_S \vec{S}\cdot\hat{n}\ dA$$

or

$$\int_V \frac{\partial}{\partial t}\frac{1}{2}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right)\,dV = -\oint_S \vec{S}\cdot\hat{n}\,dA - \int_V \vec{j}\cdot\vec{E}\,dV \tag{8}$$

where $\vec{S}$ is the Poynting vector $\vec{E}\times\vec{H}$. Remembering that $\hat{n}$ is the outward normal, we may interpret this result as:

rate of change of stored energy in the volume = flow of energy into the volume − work done by fields

Applying the divergence theorem to the first term on the RHS of (8), we may write the energy conservation equation in differential form:

$$\frac{\partial u_{\text{em}}}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{j}\cdot\vec{E} \tag{9}$$

Compare with the equation of charge conservation (Notes 1 eqn 7). Here the non-zero term on the right hand side shows that energy may be converted to non-electromagnetic forms by the currents.

1.4 Systems of conductors

With the usual convention that $\Phi \to 0$ at infinity, a spherical conductor of radius a carrying charge Q has potential $V = kQ/a$. Similarly, the surface of any conductor is an equipotential and the value of the potential is proportional to the charge the conductor carries. The constant of proportionality depends on the size and shape of the conductor:

$$V = Q/C \tag{10}$$

where C is the *capacitance*. Note that C equals the charge on the conductor when it is raised to unit potential.

Now suppose we have a collection of charged conductors. The fields produced by all of the conductors affect the potential on each of them. Thus we may write

$$V_i = \sum_{j=1}^{N} p_{ij} Q_j \tag{11}$$

The coefficients p_{ij} depend on the geometry. We can invert the matrix p_{ij} to obtain the relation

$$Q_i = \sum_{j=1}^{N} C_{ij} V_j \tag{12}$$

If all the conductors except one are grounded, and the remaining one has $V_i = 1$, then

$$Q_i = C_{ii} V_i = C_{ii}$$

Compare this relation with equation (10). The coefficients C_{ii} are called capacitances while the C_{ij} with $i \neq j$ are called coefficients of induction.

The capacitance C_{ii} is the charge required to raise conductor i to unit potential with all the others grounded.

Note that this definition is not quite the same as the capacitance of a system of two conductors defined in elementary treatments (*e.g.* Lea and Burke Ch 27 and example below). Since the name is the same, you must use context to decide which is meant.

We may also express the energy of the system in terms of the C_{ij}. Since charge exists only on the surface of the conductors (J Problem 1.1), and each surface is an equipotential, equation (3) becomes:

$$\begin{aligned}
U &= \frac{1}{2}\int_{\text{all space}} \rho(\vec{x})\,\Phi(\vec{x})\,d^3\vec{x} = \sum_{i=1}^{N}\int_{V_i} \rho(\vec{x})\,\Phi(\vec{x})\,d^3\vec{x} \\
&= \sum_{i=1}^{N}\int_{S_i} \sigma(\vec{x})\,V_i dA = \frac{1}{2}\sum_{i=1}^{N} V_i \int_{S_i} \sigma(\vec{x})\,dA \\
&= \frac{1}{2}\sum_{i=1}^{N} Q_i V_i \\
U &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij} V_j V_i = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} Q_i p_{ij} Q_j \qquad (13)
\end{aligned}$$

As an example of finding coefficents of capacitance, consider a simple system composed of two concentric, spherical conducting shells of radii a and $b > a$. Let the potential be zero at infinity, and let the charges on the two conductors be Q_a and Q_b respectively. Then we can use the spherical symmetry and Gauss' law to find the electric fields, and hence the potentials on the two surfaces.

$$\text{For } r < a, \qquad \vec{E} = 0$$

$$\text{For } a < r < b, \qquad \vec{E} = k\frac{Q_a}{r^2}\hat{r}$$

$$\text{For } r > b, \qquad \vec{E} = k\frac{Q_a + Q_b}{r^2}\hat{r}$$

To find the potentials, use the line integral (Notes 1 eqn 30):

$$\Phi_1 - \Phi_2 = \int_1^2 \vec{E}\cdot d\vec{\ell}$$

We start from infinity where we chose $\Phi = 0$. Then:

$$\begin{aligned}
\Phi_b - \Phi_\infty &= \Phi_b = k\int_b^\infty \frac{Q_a + Q_b}{r^2}dr = -k\left.\frac{Q_a + Q_b}{r}\right|_b^\infty \\
&= k\frac{Q_a + Q_b}{b}
\end{aligned}$$

and then:

$$\Phi_a - \Phi_b = k\int_a^b \frac{Q_a}{r^2}dr = -k\left.\frac{Q_a}{r}\right|_a^b = kQ_a\left(\frac{1}{a} - \frac{1}{b}\right) \tag{14}$$

and so

$$\Phi_a = k\frac{Q_a + Q_b}{b} + kQ_a\left(\frac{1}{a} - \frac{1}{b}\right) = k\left(\frac{Q_a}{a} + \frac{Q_b}{b}\right)$$

Using superposition, we can think of this as the constant potential inside the charged sphere of radius b due to its charge Q_b, plus the potential at the surface of the charged sphere of radius a due to its own charge Q_a.

Now we want to put all this into the language of capacitances. First:

$$V_i = \sum_j p_{ij} Q_j$$

or, equivalently:

$$\begin{pmatrix} V_a \\ V_b \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} Q_a \\ Q_b \end{pmatrix}$$

For our example, we can see that

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = k\begin{pmatrix} 1/a & 1/b \\ 1/b & 1/b \end{pmatrix}$$

Then we can also write:

$$Q_i = \sum_j C_{ij} V_j$$

or

$$\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} V_a \\ V_b \end{pmatrix}$$

where

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^{-1} = \frac{1}{k}\begin{pmatrix} 1/a & 1/b \\ 1/b & 1/b \end{pmatrix}^{-1} = \frac{1}{k}\left(\frac{b}{b-a}\right)\begin{pmatrix} a & -a \\ -a & b \end{pmatrix} \tag{15}$$

We may put in the expressions for the potentials to check this result:

$$\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} = \frac{1}{k}\frac{b}{b-a}\begin{pmatrix} a & -a \\ -a & b \end{pmatrix}\begin{pmatrix} V_a \\ V_b \end{pmatrix} = \frac{1}{k}\frac{b}{b-a}\begin{pmatrix} a\left(V_a - V_b\right) \\ -aV_a + bV_b \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} &= \frac{1}{k}\frac{ab}{b-a}\begin{pmatrix} V_a - V_b \\ -V_a + bV_b/a \end{pmatrix} = \frac{1}{k}\frac{ab}{b-a}k\begin{pmatrix} \frac{aQ_b+Q_ab}{ab} - \frac{Q_a+Q_b}{b} \\ -\frac{aQ_b+Q_ab}{ab} + \frac{b}{a}\frac{Q_a+Q_b}{b} \end{pmatrix} \\ &= \frac{ab}{b-a}\begin{pmatrix} Q_a\frac{b-a}{ab} \\ Q_b\frac{b-a}{ab} \end{pmatrix} = \begin{pmatrix} Q_a \\ Q_b \end{pmatrix}\end{aligned}$$

as expected.

The capacitances of the two conductors (from (15)) are

$$C_{11} = \frac{1}{k}\frac{ab}{b-a} \qquad \text{and} \qquad C_{22} = \frac{1}{k}\frac{b^2}{b-a}$$

.

The "capacitance" of this system, as defined in elementary treatments for $Q_a = -Q_b = Q$, is (equation 14)

$$C_{\text{pair}} = \left|\frac{Q}{\triangle V}\right| = \frac{Q}{kQ\,(b-a)\,/ab} = \frac{4\pi\varepsilon_0 ab}{(b-a)} = C_{11}$$

Is this relation true in general? No.

In general, if $Q_b = -Q_a$, we have:

$$\begin{pmatrix} V_a \\ V_b \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} Q \\ -Q \end{pmatrix} = \begin{pmatrix} p_{11}Q - p_{12}Q \\ p_{21}Q - p_{22}Q \end{pmatrix}$$

and thus

$$C_{\text{pair}} = \left|\frac{Q}{\Delta V}\right| = \left|\frac{1}{p_{11} - p_{12} - p_{21} + p_{22}}\right|$$

Using Green's reciprocation theorem (J problem 1.12), we can show that $p_{12} = p_{21}$, and thus

$$C_{\text{pair}} = \left|\frac{1}{p_{11} - 2p_{12} + p_{22}}\right|$$

whereas

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^{-1} = \frac{1}{p_{11}p_{22} - p_{12}p_{21}} \begin{pmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{pmatrix}$$

So

$$C_{11} = \frac{p_{22}}{p_{11}p_{22} - p_{12}p_{21}} = \frac{p_{22}}{p_{11}p_{22} - p_{12}^2} = \frac{1}{p_{11} - p_{12}^2/p_{22}}$$

which is not equal to C_{pair}, in general.

For *this* system of two spheres, $p_{22} = p_{12}$ (you might want to investigate the conditions necessary for this relation to hold), and we have:

$$C_{11} = \frac{1}{p_{11} - p_{22}} = C_{\text{pair}}$$

The energy of the system is:

$$\begin{aligned} U &= \frac{1}{2} Q \cdot P \cdot Q \\ &= \frac{1}{2} \begin{pmatrix} Q_a & Q_b \end{pmatrix} k \begin{pmatrix} 1/a & 1/b \\ 1/b & 1/b \end{pmatrix} \begin{pmatrix} Q_a \\ Q_b \end{pmatrix} \\ &= \frac{k}{2}\left(\frac{Q_a^2}{a} + 2\frac{Q_a}{b}Q_b + \frac{Q_b^2}{b}\right) \end{aligned}$$

Note that we could also get this result from $Q_a\Phi_a + Q_b\Phi_b$. In the special case that $Q_a = -Q_b = Q$, we get

$$U = \frac{k}{2}\left(\frac{Q^2}{a} - 2\frac{Q^2}{b} + \frac{Q^2}{b}\right) = \frac{kQ^2}{2}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{2}\frac{Q^2}{C_{\text{pair}}} \tag{16}$$

as expected.

1.5 Poynting's theorem for harmonic fields

The above discussion does not include any dissipative effects that convert energy to non-electromagnetic forms. The easiest way to include these effects is by working with the Fourier transforms of the Maxwell equations. This immediately gives us the results for AC circuits, for example, by considering a single frequency component. The general results for time dependent fields with dissipation are in Jackson §6.8.

For a single frequency component

$$\vec{E}\left(\vec{x},t\right) = \text{Re}\left(\vec{E}\left(\vec{x},\omega\right)e^{-i\omega t}\right) = \frac{1}{2}\left(\vec{E}\left(\vec{x},\omega\right)e^{-i\omega t} + \vec{E}^*\left(\vec{x},\omega\right)e^{i\omega t}\right)$$

When multiplying two such terms, we have to be careful. For example, the rate at which the fields do work is

$$\begin{aligned}
P\left(\vec{x},t\right) &= \text{Re}\left[\vec{j}\left(\vec{x},t\right)\right]\cdot\text{Re}\left[\vec{E}\left(\vec{x},t\right)\right] \\
&= \frac{1}{2}\left(\vec{j}\left(\vec{x},\omega\right)e^{-i\omega t} + \vec{j}^*\left(\vec{x},\omega\right)e^{i\omega t}\right)\frac{1}{2}\left(\vec{E}\left(\vec{x},\omega\right)e^{-i\omega t} + \vec{E}^*\left(\vec{x},\omega\right)e^{i\omega t}\right) \\
&= \frac{1}{4}\Big[\vec{j}\left(\vec{x},\omega\right)\cdot\vec{E}^*\left(\vec{x},\omega\right) + \vec{j}^*\left(\vec{x},\omega\right)\cdot\vec{E}\left(\vec{x},\omega\right) + \vec{j}\left(\vec{x},\omega\right)\cdot\vec{E}\left(\vec{x},\omega\right)e^{-2i\omega t} \\
&\qquad + \vec{j}^*\left(\vec{x},\omega\right)\cdot\vec{E}^*\left(\vec{x},\omega\right)e^{2i\omega t}\Big] \\
&= \frac{1}{2}\text{Re}\left(\vec{j}^*\left(\vec{x}\right)\cdot\vec{E}\left(\vec{x}\right) + \vec{j}\left(\vec{x}\right)\cdot\vec{E}\left(\vec{x}\right)e^{-2i\omega t}\right)
\end{aligned}$$

Taking the time average, the second term vanishes, and we get

$$< P\left(\vec{x}\right) > \; = \frac{1}{2}\text{Re}\left[\vec{j}^*\left(\vec{x}\right)\cdot\vec{E}\left(\vec{x}\right)\right]$$

In what follows, the "Re" will be implicit. Then with these conventions, the transformed Maxwell's equations become

$$\begin{aligned}
\vec{\nabla}\cdot\vec{D} &= \rho_{\text{f}} \\
\vec{\nabla}\cdot\vec{B} &= 0 \\
\vec{\nabla}\times\vec{E} &= i\omega\vec{B} \\
\vec{\nabla}\times\vec{H} + i\omega\vec{D} &= \vec{j}_f
\end{aligned}$$

and the time-averaged rate at which the fields do work in a volume V is

$$\begin{aligned} < \quad P > \quad &= \frac{1}{2}\int \vec{j}^*(\vec{x})\cdot\vec{E}(\vec{x})\ dV \\ &= \frac{1}{2}\int \left[\vec{\nabla}\times\vec{H}^* - i\omega\vec{D}^*\right]\cdot\vec{E}(\vec{x})\ dV \end{aligned}$$

But

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}^*\right) = \vec{H}^*\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{H}^*\right)$$

So

$$\begin{aligned} < \quad P > \quad &= \frac{1}{2}\int \left[-\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}^*\right) + \vec{H}^*\cdot\left(\vec{\nabla}\times\vec{E}\right) - i\omega\vec{D}^*\cdot\vec{E}(\vec{x})\right]\ dV \\ &= \frac{1}{2}\int \left[-\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}^*\right) + \vec{H}^*\cdot\left(i\omega\vec{B}\right) - i\omega\vec{D}^*\cdot\vec{E}(\vec{x})\right]\ dV \end{aligned}$$

The time averaged rate of energy flow is

$$\vec{S} = \frac{1}{2}\left(\vec{E}\times\vec{H}^*\right) = \frac{1}{2}\left(\vec{E}^*\times\vec{H}\right)$$

making

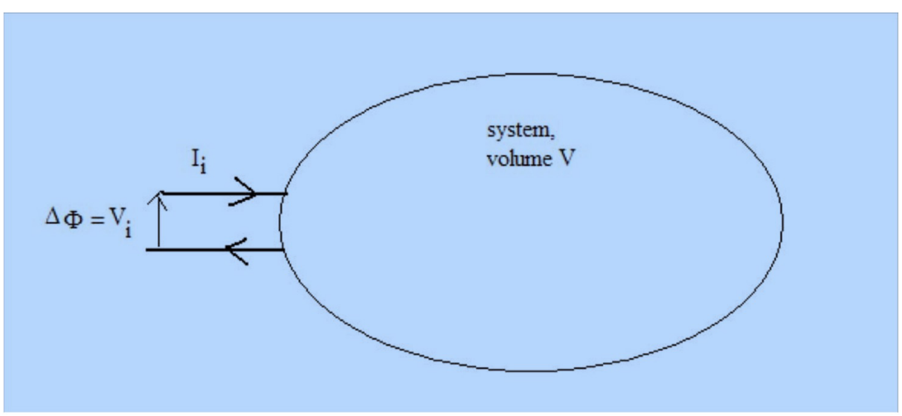
$$< \quad P > \quad = -\oint_S \vec{S}\cdot n\ dA - i\frac{\omega}{2}\int \left[\vec{D}^*\cdot\vec{E} - \vec{H}^*\cdot\vec{B}\right]\ dV$$

$$\frac{1}{2}\int \vec{j}^*(\vec{x})\cdot\vec{E}(\vec{x})\,dV \quad = \quad -\oint_S \vec{S}\cdot\hat{n}\ dA - 2i\omega\int (u_E - u_M)\ dV \qquad (17)$$

Note that the last term has a real part only if

$$u_E = \frac{1}{4}\left(\vec{D}^*\cdot\vec{E}\right) = \frac{\varepsilon^*(\omega)}{4}\left|\vec{E}\right|^2$$

has an imaginary part, that is, ε has an imaginary part, indicating losses. $\mathrm{Re}\,(u_E)$ is the time-averaged electric energy density at frequency ω, and simlarly for u_M. Equation (17) is the analog of equation (8) for harmonic fields.



Now consider a system with all the fields confined inside a volume V, as shown. The Poynting flux $\vec{S}$ is zero on the bounding surface S. (This means we have no radiation resistance. See J §6.9 for a fuller discussion that includes radiation resistance). Instead power is provided to the system by an input current I_i. V_i is the potential difference across the system, input terminal to output terminal, as shown in the diagram. Then we modify equation (17) as follows:

$$\frac{1}{2}I_i^* V_i = \frac{1}{2}\int \vec{j}\,(\vec{x})^* \cdot \vec{E}\,(\vec{x})\,dV + 2i\omega\int (u_E - u_M)\ dV$$

We can interpret this in terms of the impedance of the system (see, eg, Lea Ch 2 Prob 14, Jackson page 266):

$$V_i = I_i Z = I_i\,(R - iX)$$

where R is the resistance and X is the reactance. (The minus sign is a result of choosing the time dependence to be $e^{-i\omega t}$.) Then, if the conductivity is real,

$$\begin{aligned}
|I_i|^2\,Z &= \int \vec{j}\,(\vec{x})^* \cdot \vec{E}\,(\vec{x})\,dV + 4i\omega\int (u_E - u_M)\ dV \\
R - iX &= \frac{1}{|I_i|^2}\left\{\int \sigma\left|\vec{E}\right|^2 dV + 4i\omega\int (u_E - u_M)\ dV\right\}
\end{aligned}$$

Thus, equating real and imaginary parts, we have

$$R = \frac{1}{|I_i|^2}\int \sigma\left|\vec{E}\right|^2 dV$$

and

$$X = -\frac{4\omega}{|I_i|^2}\int (u_E - u_M)\ dV \tag{18}$$

At low frequencies, the time-averaged stored energies may be expressed in terms of capacitance (eqn 16) and inductance as

$$U_E = \frac{1}{4}\frac{|Q|^2}{C} = \frac{1}{4}\frac{|I_i|^2}{\omega^2 C} \text{ and } U_M = \frac{1}{4}L\,|I_i|^2$$

where $I_i = \frac{dQ}{dt} = -i\omega Q$, so that (18) reduces to

$$X = \omega\left(L - \frac{1}{\omega^2 C}\right) = \omega L - \frac{1}{\omega C}$$

as expected.