MAGNETIC FIELDS IN MATTER

Magnetization

When considering the electric fields in matter we needed to think about the atomic structure of the material. Applied electric fields tend to distort the atoms and/or molecules by stretching and rotating them. The electron orbits in atoms constitute little current loops, and so they form little magnetic dipoles. As electric fields align electric dipoles with a torque $\vec{\tau} = \vec{p} \times \vec{E}$, so also magnetic fields align magnetic dipoles with a torque $\vec{\tau} = \vec{m} \times \vec{B}$. (This is how compass needles work.) In most materials, thermal agitation ensures that the atomic dipoles are randomly oriented, but applied fields tend to align the dipoles parallel to $\vec{B}$. The material is then magnetized. However, the result of the alognment is quite different from what we found with electric fields. The electric dipoles create a charge layer on the surface of a dielectric, and that layer produces a field within the material that is opposite the applied field. Thus the result is a net field inside the material that is smaller than the applied field.

Aligned magnetic dipoles produce a surface current layer (LB Figure 29.25) and the magnetic field inside this current loop is parallel to the applied field. Thus the net result is a field inside the material that is greater than the applied field. The effect is called paramagnetism.

But this is not yet the whole story. Changing magnetic fields produce electric fields through Faraday's law, and those electric fields effect the atomic currents. Let's see how it goes.

We model a simple atomic magnetic moment as an electron orbiting with radius r at speed v. The magnetic moment is

$$m = IA = \frac{e}{T}\pi r^2$$

where T is the period $2\pi r/v$. Thus

$$m = \frac{e}{2}rv$$

Now let's worry about directions. The current is opposite the velocity, because the charge is negative, and so if $\vec{\omega} = \omega\hat{z} = \frac{v\hat{z}}{r}$, then

$$\vec{m} = -\frac{e}{2}rv\hat{z}$$

The flux of external magnetic field through this loop is

$$\Phi_{\text{ext}} = \pi r^2 \vec{B}_{\text{ext}} \cdot \hat{z}$$

and the changing flux produces an emf

$$\varepsilon = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_{\text{ext}}}{dt} = -\pi r^2 \frac{d}{dt}\vec{B}_{\text{ext}} \cdot \hat{z}$$

If we start with $\vec{B}_{\text{ext}} = 0$, the centripetal force on the electron orbit is purely electric:

$$\vec{F}_c = -\frac{mv^2}{r}\hat{r} = -e\frac{Ze}{4\pi\varepsilon_0 r^2}\hat{r} \tag{1}$$

The induced electric field is given by

$$2\pi r E_\phi = -\pi r^2 \frac{d}{dt} \vec{B}_{\text{ext}} \cdot \hat{z}$$

So if $\vec{B} \cdot \hat{z}$ is increasing, E_θ will be negative, F_ϕ will be positive and the particle's speed will increase. However, if $\vec{B} \cdot \hat{z}$ is decreasing, E_θ will be positive, F_ϕ will be negative and the particle's speed will decrease. So after time Δt, the speed will have changed by Δv, and we also have an additional force component $\vec{F}_{\text{mag}} = -e\vec{v} \times \vec{B}$

$$F_r = -\frac{Ze^2}{4\pi\varepsilon_0 r} - evB_z = -\frac{m\left(v+\Delta v\right)^2}{r} = -\frac{mv^2}{r} - \frac{2mv\Delta v}{r}$$

where we ignore the square of the small change Δv, as usual. The first terms on each side are equal, by (1). Thus

$$\begin{aligned} evB_z &= \frac{2mv\Delta v}{r} \\ \Delta v &= \frac{eB_z r}{2m} \end{aligned}$$

As we concluded above, Δv is positive if B_z is positive. This change in speed also changes the magnetic moment:

$$\vec{m} + \Delta\vec{m} = -\frac{e}{2} r \left(v + \Delta v\right) \hat{z}$$

So

$$\Delta\vec{m} = -\frac{er}{2}\left(\frac{eB_z r}{2m}\right)\hat{z} = -\frac{e^2 r^2}{4m}\vec{B}_{\text{ext}}$$

The change in $\vec{m}$ is opposite the change in $\vec{B}$. That means that increasing the external $\vec{B}$ tends to decrease the internal field due to the dipoles. This effect is called diamagnetism.

Both these effects happen at the same time. Diamagnetic effects are usually weak, and show up in materials in which the intrinsic atomic magnetic moments are small (these are usually atoms with an even number of electrons, like Helium) or materials that are hot, because the thermal agitation dirupts the alignment of the dipoles. What this means is that the response of any given material to an applied magnetic field will depend on that material's properties: in some cases the internal field is increased (these are paramagnetic materials) and in others, decreased (these are diamagnetic materials). In both cases the effect is small.

A very interesting class of materials, the ferromagnetic materials, are able to maintain the alignment of the dipoles even when the applied field is removed, thus forming a permanent magnet. The details are due to quantum mechanical interactions, and we won't discuss them here. Each ferrormagnetic material has a characteristic temperature, called the Curie temperature, Above that temperature, thermal effects disrupt the alignment and a permanent magnet cannot be formed. There are additional very interesting features of these materials, such as hysteresis, which make them more difficult to analyze. More later.

As with dielectric polarization, we define the magnetization of a magnetic material $\vec{M}$ as the dipole moment per unit volume,

$$\vec{M} = n\vec{m}$$

Dielectrics are drawn into stronger electric field regions. Similarly, paramagnetic materials are drawn into stronger magnetic field regions, while diamagnetic materials are repelled from such regions.

Fields in magnetic materials.

With dielctrics we found it useful to introduce a new field $\vec{D}$ to aid in our analysis. Similarly here we shall find it useful to introduce a new field $\vec{H}$. The proper name for $\vec{B}$ is magnetic induction, while $\vec{H}$ is called the magnetic field. (Griffiths seems to hate this language, and I do too, so for the most part we'll ignore it, and continue to call $\vec{B}$ the magnetic field. Please use context to be sure you know what is meant.) So let's see how this goes.

We start with the magnetic vector potential due to one dipole.

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

and from this we can compute $\vec{A}$ due to a magnetized object:

$$\vec{A}\left(\vec{r}\right) = \frac{\mu_0}{4\pi}\int\frac{\vec{M}\times\hat{R}}{R^2}d\tau'$$

where, as usual, $\vec{R} = \vec{r} - \vec{r}'$. We may rewrite the integrand as follows:

$$\begin{aligned}
\vec{A}\left(\vec{r}\right) &= \frac{\mu_0}{4\pi}\int\vec{M}\times\left(\vec{\nabla}'\frac{1}{R}\right)d\tau' \\
&= \frac{\mu_0}{4\pi}\int\left[-\vec{\nabla}'\times\left(\frac{\vec{M}}{R}\right)+\frac{\vec{\nabla}'\times\vec{M}}{R}\right]d\tau' \\
&= \frac{\mu_0}{4\pi}\left[\int_s\frac{\vec{M}}{R}\times d\vec{A}' + \int\frac{\vec{\nabla}'\times\vec{M}}{R}d\tau'\right] \qquad (2)
\end{aligned}$$

We may identify the bound current density

$$\vec{j}_{\text{bound}} = \vec{\nabla}\times\vec{M} \qquad (3)$$

and bound surface current density

$$\vec{K}_{\text{bound}} = \vec{M}\times\hat{n} \qquad (4)$$

so that $\vec{A}$ becomes

$$\vec{A}\left(\vec{r}\right) = \frac{\mu_0}{4\pi}\left[\int_s\frac{\vec{K}_{\text{bound}}}{R}dA' + \int\frac{\vec{j}_{\text{bound}}}{R}d\tau'\right]$$

Equation (2) may be used for any magnetized object, including permanent magnets.

Suppose we have a uniformly magnetized sphere, with $\vec{M} = M\hat{z}$ inside, and zero outside, of course. Then $\vec{j}_{\text{bound}} = 0$, because $\vec{M}$ is constant except *at* the surface. But we take care of the surface with $\vec{K}$.

$$\vec{K}_{\text{bound}} = \vec{M} \times \hat{r} = M \sin\theta\ \hat{\phi}$$

Now this looks just like our rotating, uniformly charged spherical shell, and the field is identical if we replace $\sigma\omega a$ with M. The internal field is uniform and the external field is a pure dipole.

$$\vec{B}_{\text{int}} = \frac{2}{3}\mu_0\vec{M}$$

and the dipole moment is

$$\vec{m} = \frac{4\pi}{3}a^3\vec{M} = \vec{M} \times \text{volume of sphere}$$

This exactly what we might have expected.

We can use this result to motivate our expression for $\vec{K}_{\text{bound}}$. For any volume V,

$$\vec{m} = \vec{M}V$$

Let us look at a slab of area A and thickness t, so that

$$m = MAt = IA$$

Thus the effective "loop"current is $I = Mt$ and thus the surface current is $K = I/t = M$. Putting back the vectors to get the directions gives equation (4).

Now we take the total current and divide it into two parts: the bound current due to magnetization and the free current due to moving charges,

$$\vec{j} = \vec{j}_{\text{bound}} + \vec{j}_{\text{free}}$$

Ampere's law in differential form is

$$\vec{\nabla} \times \vec{B} = \mu_0\left(\vec{j}_{\text{bound}} + \vec{j}_{\text{free}}\right) = \mu_0\left[\left(\vec{\nabla} \times \vec{M}\right) + \vec{j}_{\text{free}}\right]$$

Put the curl terms together to get

$$\vec{\nabla} \times \left(\vec{B} - \mu_0\vec{M}\right) = \mu_0\vec{j}_{\text{free}}$$

Now define the field $\vec{H}$ by

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{5}$$

and we get

$$\vec{\nabla} \times \vec{H} = \vec{j}_{\text{free}} \tag{6}$$

We can then deduce the integral form of Ampere's law for $\vec{H}$, which is

$$\oint \vec{H} \cdot d\vec{l} = I_{\rm free} = \int \vec{j}_{\rm free} \cdot \hat{n}\ dA$$

This is a very useful relation in situations with sufficient symmetry.

$\vec{H}$ satisfies a second equation that we can get from its definition (5). Taking the divergence, we get

$$\vec{\nabla} \cdot \vec{H} \equiv \frac{1}{\mu_0}\vec{\nabla} \cdot \vec{B} - \vec{\nabla} \cdot \vec{M}$$

But $\vec{\nabla} \cdot \vec{B} = 0$, so

$$\vec{\nabla} \cdot \vec{H} \equiv -\vec{\nabla} \cdot \vec{M} \tag{7}$$

This can be very useful in situations where $\vec{j}_{\rm free} = 0$. For then equation (6) becomes $\vec{\nabla} \times \vec{H} = 0$, and so we may express $\vec{H}$ as the gradient of a scalar function I'll call χ. If we write

$$\vec{H} = -\vec{\nabla}\chi$$

equation (7) becomes Poisson's equation for χ.

$$\nabla^2\chi = \vec{\nabla} \cdot \vec{M}$$

This problem is exactly analgous to an electrostatic potential problem, where $-\vec{\nabla} \cdot \vec{M}$ acts like a magnetic charge density.

Now we can see some important differences between $\vec{B}$ and $\vec{H}$. The field lines of $\vec{B}$ form closed loops, because $\vec{\nabla} \cdot \vec{B} = 0$. But the field lines of $\vec{H}$ begin and end on "magnetic charge" where $\vec{\nabla} \cdot \vec{M} \neq 0$.

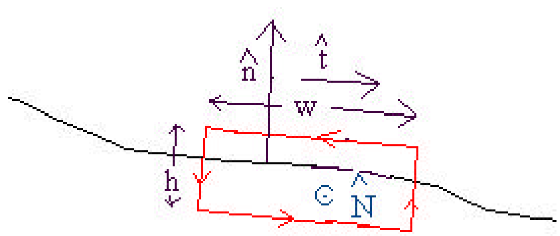
Let's look again at the magnetized sphere. Inside the sphere

$$\vec{H}_{\rm in} = \frac{1}{\mu_0}\vec{B} - \vec{M} = \frac{1}{\mu_0}\frac{2}{3}\mu_0\vec{M} - \vec{M} = -\frac{\vec{M}}{3}$$

Thus the lines of $\vec{H}$ diverge from the surface of the sphere where $\vec{\nabla} \cdot \vec{M} \neq 0$.

Boundary conditions for $\vec{H}$

Starting with the curl equation, and integrating around a rectangle across the boundary, we have



$$\begin{aligned}
\int\left(\vec{\nabla}\times\vec{H}\right)\cdot d\vec{A} &= \int\vec{j}_{\text{free}}\cdot dA \\
\oint\vec{H}\cdot d\vec{l} &= \hat{N}\cdot\int\vec{j}_{\text{free}}dh\; w \\
\left(\vec{H}_1-\vec{H}_2\right)\cdot\left(-\hat{t}\right) &= \hat{N}\cdot\vec{K}_{\text{free}}
\end{aligned}$$

Since $\hat{t}=\hat{n}\times\hat{N}$, we get

$$\hat{n}\times\left(\vec{H}_1-\vec{H}_2\right)=\vec{K}_{\text{free}} \tag{8}$$

Then integrating (7) over a pillbox, we get

$$\begin{aligned}
\int\vec{\nabla}\cdot\vec{H}dV &\equiv -\int\vec{\nabla}\cdot\vec{M}dV \\
\int\vec{H}\cdot d\vec{A} &= -\int\vec{M}\cdot d\vec{A}
\end{aligned}$$

Now if $\vec{M}$ is zero on one side of the surface (a frequent occurrence) we get

$$\left(\vec{H}_{\text{out}}-\vec{H}_{\text{in}}\right)\cdot\hat{n}=\vec{M}\cdot\hat{n} \tag{9}$$

$\vec{M}\cdot\bar{n}$ acts like a surface "magnetic charge density" in producing $\vec{H}$.

Suppose a bar magnet has a uniform magnetization $\vec{M}$ running along its length. Then $\vec{\nabla}\cdot\vec{M}$ is zero everywhere except at the surface of the magnet (where it is infinite). $\vec{M}\cdot\hat{n}$ is also zero except at the ends of the magnet. Thus we have no "magnetic charge density" except at the two ends, where $\vec{M}\cdot\hat{n}$ is positive on one end and negative on the other. Thus we have a physical dipole. Inside the magnet $\vec{H}$ runs from the positive "magnetic charge" at one end to the negative "magnetic charge" on the other. Outside we have a dipole-type field.

Now $\vec{B}=\mu_0\vec{H}+\vec{M}$ forms closed loops. Outside there is no $\vec{M}$ and the $\vec{B}$ field lines follow the $\vec{H}$ lines exactly. But the $\vec{B}$ lines form closed loops that close inside the magnet, where $\vec{B}$ and $\vec{H}$ are opposite each other.

Let's summarize the boundary conditions for $\vec{B}$ and $\vec{H}$:

$$\begin{aligned}
& B_{\text{normal}}\text{ is continuous always} \\
\hat{n}\times\left(\vec{H}_1-\vec{H}_2\right) &= \vec{K}_{\text{free}};\quad \vec{H}_{\text{tangential}}\text{ is continuous if }\vec{K}_{\text{free}}=0
\end{aligned}$$

Linear Media

In LIH materials, the magnetization is proportional to the net field:

$$\vec{M}=\chi_m\vec{H}$$

Here χ_m is the magnetic susceptibility. Unlike the electric susceptibility, χ_m can take either sign: it is positive in paramagnetic materials and negative in diamagnetic materials. But in both cases it is almost always very small, $\chi_m \ll 1$. For LIH materials,

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) = \mu_0 \left(1 + \chi_m\right) \vec{H} = \mu \vec{H}$$

μ is the permeability of the material. For most LIH materials where this formulation makes sense, $\mu \simeq \mu_0$.

For these materials the bound current is

$$\vec{j}_{\text{bound}} = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times \left(\chi_m \vec{H} \right)$$

and because of the IH part of LIH, we can move the susceptibilty through the derivative to get

$$\vec{j}_{\text{bound}} = \chi_m \vec{\nabla} \times \vec{H} = \chi_m \vec{j}_{\text{free}}$$

If there is no free current density inside the material, all the bound current will be at the surface.

Feromagnetism

In a ferromagnetic material, the atomic dipoles tend to align with each other, even in the absence of applied fields. the dipoles align in regions called domains. Normally the domains are oriented randomly. When an external field is applied, the dipoles at the edge of a domain feel an additonal torque, and the net effect is to cause the domains with magnetization parallel to the applied field to grow. If the applied field is strong enough, the other domains actually rotate to align with the applied field. When all the domains are aligned, the material is saturated and $\vec{B}$ has its maximum value.

To magnetize the material we set up a coil so that we get an (almost) uniform $\vec{H}$. (Remember it is $\vec{H}$ that is directly related to the free currents.) As the current is increased, H increases and so does the magnetization. But once the material is saturated, M does not increase further. Then B can increase only slowly, since

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

with $M \gg H$. If we now begin to turn the current (and thus H) down, the domains remain aligned Only when $\vec{H}$ reverses do we get a substantial decrease in $\vec{M}$. Thus the relation between $\vec{B}$ and $\vec{H}$ is not at all linear. Worse, the value of $\vec{B}$ does not depend only on $\vec{H}$, but also on the history– what has happened previously. This phenomenon is called hysteresis. The graph of B versus H looks like this:

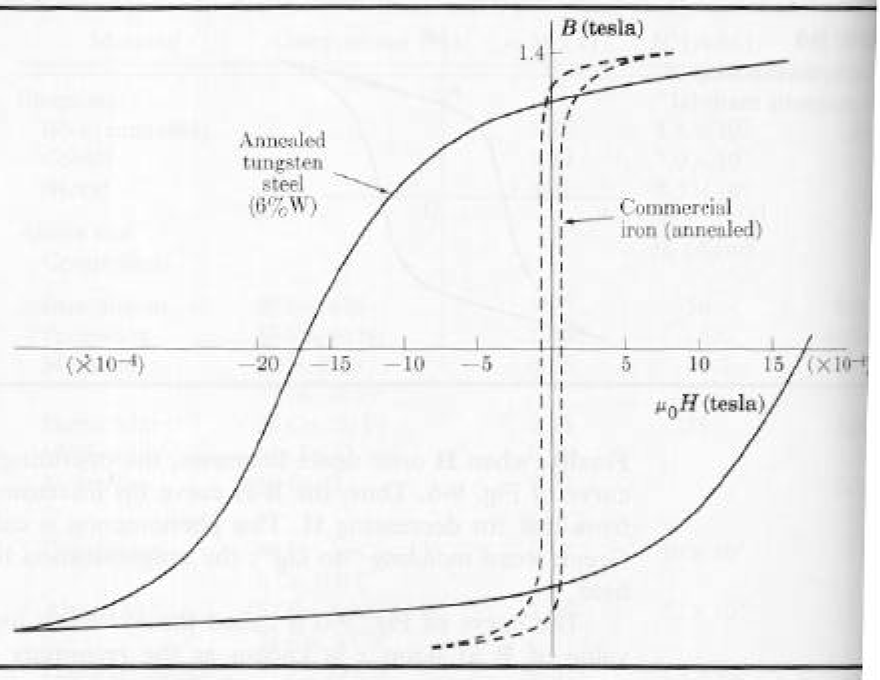


FIGURE 9-7 Comparison of the hysteresis curves of several materials. (Note that $\mu_0 H$ is plotted along the abscissa instead of just H. $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$.) Data from R. M. Bozorth, *Ferromagnetism* (New York: Van Nostrand, 1951).

This is a hysteresis loop (Diagram from Reitz,Milford and Christy). . (See LB page 942 for more examples.)

There are two kinds of ferromagnetic materials. Materials with a fat hysteresis loop are hard magnetic materials that are used to make permanent magnets. Such materials include cobalt steel and the alloy called alnico. Materials that have a thin hysteresis loop (such as ferrites) are soft magnetic materials used in transformer cores. A H value of about 80 A/m is sufficient to saturate iron, which is a soft magnetic material.

Magnetic shielding

One of the important uses of magnetic materials is for magnetic shielding–creating a volume of space where magnetic fields are reduced to almost zero.

As an example, consider an infinitely long cylindrical shell with internal and external radii a and b, and large magnetic permeability μ. We place the $z-$axis along the axis of the cylinder. The shell is placed in a uniform external field $\vec{B}_0 = B_0\hat{x}$. Let's find the magnetic field everywhere.

We may express $\vec{H}$ as the gradient of a scalar potential χ that satisfies Laplace's equation everywhere outside the shell, and in the hole. In the shell itself

$$\vec{\nabla} \cdot \vec{H} \equiv -\vec{\nabla} \cdot \vec{M}$$

But

$$\vec{B} = \mu\vec{H}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0 = \vec{\nabla} \cdot \left(\mu \vec{H} \right) = \mu \vec{\nabla} \cdot \vec{H}$$

So

$$\nabla^2 \chi = 0$$

everywhere except at the edges of the shell.

We may express the solution in cylindrical coordinates with no $z-$dependence. Now we write three solutions, one valid in each of the three regions. In the hole, the solution must be finite as $r \to 0$, so

$$\chi_{\text{in}} = \sum_{n=1}^{\infty} r^n \left(A_n \sin n\phi + B_n \cos n\phi \right)$$

In the exterior, the solution must give us the applied field as $r \to \infty$, so

$$\chi_{\text{out}} = -\frac{B_0}{\mu_0} r \cos \phi + \sum_{n=1}^{\infty} r^{-n} \left(C_n \sin n\phi + D_n \cos n\phi \right)$$

In the shell itself, we need all the terms:

$$\chi_{\text{shell}} = \sum_{n=1}^{\infty} \left(r^{-n} + \alpha_n r^n \right) \left(E_n \sin n\phi + F_n \cos n\phi \right)$$

Now we apply the boundary conditions.

Normal B is continuous at each boundary.

$$-B_0 \cos \phi - \mu_0 \sum_{n=1}^{\infty} n b^{-n-1} \left(C_n \sin n\phi + D_n \cos n\phi \right) = \mu \sum_{n=1}^{\infty} \left(-n b^{-n-1} + n \alpha_n b^{n-1} \right) \left(E_n \sin n\phi + F_n \cos n\phi \right)$$

and

$$\mu_0 \sum_{n=1}^{\infty} n a^{n-1} \left(A_n \sin n\phi + B_n \cos n\phi \right) = \mu \sum_{n=1}^{\infty} \left(-n a^{-n-1} + n \alpha_n a^{n-1} \right) \left(E_n \sin n\phi + F_n \cos n\phi \right)$$

Tangential $\vec{H}$ is continuous at the boundaries:

$$\frac{B_0}{\mu_0} b \sin \phi + \sum_{n=1}^{\infty} n b^{-n} \left(C_n \cos n\phi - D_n \sin n\phi \right) = \sum_{n=1}^{\infty} n \left(b^{-n} + \alpha_n b^n \right) \left(E_n \cos n\phi - F_n \sin n\phi \right)$$

and

$$\sum_{n=1}^{\infty} n \left(a^{-n} + \alpha_n a^n \right) \left(E_n \cos n\phi - F_n \sin n\phi \right) = \sum_{n=1}^{\infty} n a^n \left(A_n \cos n\phi - B_n \sin n\phi \right)$$

We have seen this kind of thing before. The external field imposes the $n = 1$, cosine mode– all the others are zero.Verify this for yourself. So looking at $n = 1$, we have the 4 equations:

$$-B_0 - \mu_0 b^{-2} D_1 = \mu \left(-b^{-2} + \alpha_1 \right) F_1 \tag{10}$$

$$\mu_0 B_1 = \mu\left(-a^{-2}+\alpha_1\right)F_1 \tag{11}$$

$$\frac{B_0}{\mu_0}b + b^{-1}\left(-D_1\right) = \left(b^{-1}+\alpha_1 b\right)\left(-F_1\right) \tag{12}$$

and

$$\left(a^{-1}+\alpha_1 a\right)\left(-F_1\right) = -aB_1 \tag{13}$$

for the four unknowns $B_1, D_1,\ F_1$ and α_1. We can already see from equation (10) that the coefficient F_1 has to be of order μ_0/μ. and from (13) B_1 will be of roder F_1. This is the source of the shielding.

Going at it systematically, we get from (13)

$$B_1 = \left(a^{-2}+\alpha_1\right)F_1$$

Then putting this result into (11) we have

$$\begin{aligned}
\mu\left(-a^{-2}+\alpha_1\right)F_1 &= \mu_0\left(a^{-2}+\alpha_1\right)F_1\\
\frac{\mu}{\mu_0}\left(-a^{-2}+\alpha_1\right) &= \left(a^{-2}+\alpha_1\right)\\
\alpha_1\left(\frac{\mu}{\mu_0}-1\right) &= a^{-2}\left(1+\frac{\mu}{\mu_0}\right)\\
\alpha_1 &= a^{-2}\frac{\mu+\mu_0}{\mu-\mu_0} \qquad (14)
\end{aligned}$$

Then putting (14) into (10) we have

$$-B_0 b^2 - \mu_0 D_1 = \mu\left(-1+\frac{b^2}{a^2}\frac{\mu+\mu_0}{\mu-\mu_0}\right)F_1$$

and puting (14) into (12) we get

$$B_0 b^2 - \mu_0 D_1 = -\mu_0\left(1+\frac{b^2}{a^2}\frac{\mu+\mu_0}{\mu-\mu_0}\right)F_1$$

Subtracting these two, we get

$$2B_0 b^2 = \left(\mu-\mu_0-\frac{b^2}{a^2}\frac{\left(\mu+\mu_0\right)^2}{\mu-\mu_0}\right)F_1$$

$$F_1 = \frac{2B_0 a^2 b^2\left(\mu-\mu_0\right)}{a^2\left(\mu-\mu_0\right)^2 - b^2\left(\mu+\mu_0\right)^2} \tag{15}$$

Adding gives

$$2\mu_0 D_1 = F_1\left(\mu+\mu_0\right)\left(1-\frac{b^2}{a^2}\right)$$

so

$$\begin{aligned}
D_1 &= \frac{\left(\mu+\mu_0\right)}{2\mu_0}\left(1-\frac{b^2}{a^2}\right)\frac{2B_0 a^2 b^2\left(\mu-\mu_0\right)}{a^2\left(\mu-\mu_0\right)^2-b^2\left(\mu+\mu_0\right)^2}\\
&= \frac{B_0}{\mu_0}\left(a^2-b^2\right)\frac{b^2\left(\mu^2-\mu_0^2\right)}{a^2\left(\mu-\mu_0\right)^2-b^2\left(\mu+\mu_0\right)^2} \qquad (16)
\end{aligned}$$

and finally

$$\begin{aligned} B_1 &= \left(a^{-2}+\alpha_1\right) F_1 \\ &= \frac{1}{a^2}\left(\frac{2\mu}{\mu-\mu_0}\right)\frac{2B_0 a^2 b^2\left(\mu-\mu_0\right)}{a^2\left(\mu-\mu_0\right)^2-b^2\left(\mu+\mu_0\right)^2} \\ &= \frac{4\mu B_0 b^2}{a^2\left(\mu-\mu_0\right)^2-b^2\left(\mu+\mu_0\right)^2} \end{aligned} \tag{17}$$

Thus

$$\chi_{\text{in}} = \frac{4\mu B_0 b^2}{a^2\left(\mu-\mu_0\right)^2-b^2\left(\mu+\mu_0\right)^2} r\cos\phi$$

$$\chi_{\text{out}} = \frac{B_0}{\mu_0}\cos\phi\left[\frac{\left(a^2-b^2\right)}{r}\frac{b^2\left(\mu^2-\mu_0^2\right)}{a^2\left(\mu-\mu_0\right)^2-b^2\left(\mu+\mu_0\right)^2}-r\right]$$

$$\chi_{\text{shell}} = B_0\left(\frac{\left(\mu-\mu_0\right)}{r}+\left(\mu+\mu_0\right)\frac{r}{a^2}\right)\frac{2a^2b^2}{a^2\left(\mu-\mu_0\right)^2-b^2\left(\mu+\mu_0\right)^2}\cos\phi$$

Verify that we get back $\vec{B}=\vec{B}_0$ everywhere if $\mu\to\mu_0$.

Let's see what happens as μ/μ_0 becomes very large:

$$\chi_{\text{in}} \to \frac{B_0}{\mu}\frac{4b^2}{a^2-b^2} r\cos\phi$$

$$\chi_{\text{out}} \to \frac{B_0}{\mu_0}\left[\frac{b^2}{r}-r\right]\cos\phi$$

$$\chi_{\text{shell}} \to \frac{B_0}{\mu}\left(\frac{1}{r}+\frac{r}{a^2}\right)\frac{2a^2b^2}{a^2-b^2}\cos\phi$$

The fields in the cavity become

$$\vec{H}_{\text{in}} = -\vec{\nabla}\chi_{\text{in}} \to \frac{B_0}{\mu}\frac{4b^2}{b^2-a^2}\hat{x}$$

and

$$\vec{B}_{\text{in}} = \mu_0\vec{H}_{\text{in}} \to \frac{\mu_0}{\mu}B_0\frac{4b^2}{b^2-a^2}\hat{x}$$

Both are very small, so the shell has effectively shielded the cavity from the external fields. Within the shell

$$\vec{H}_{\text{shell}} = \frac{B_0}{\mu}\frac{2b^2}{b^2-a^2}\left(\hat{x}-a^2\frac{\cos\phi\ \hat{r}+\sin\phi\ \hat{\phi}}{r^2}\right)$$

which is small, but

$$\vec{B}_{\text{shell}} = B_0\frac{2b^2}{b^2-a^2}\left(\hat{x}-a^2\frac{\cos\phi\ \hat{r}+\sin\phi\ \hat{\phi}}{r^2}\right)$$

which is comparable to B_0.