Magnetic vector potential

When we derived the scalar electric potential we started with the relation $\vec{\nabla} \times \vec{E} = 0$ to conclude that $\vec{E}$ could be written as the gradient of a scalar potential. That won't work for the magnetic field (except where $\vec{j} = 0$), because the curl of $\vec{B}$ is not zero in general. Instead, the divergence of $\vec{B}$ is zero. That means that $\vec{B}$ may be written as the curl of a vector that we shall call $\vec{A}$.

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A} \right) = 0$$

Then the second equation becomes

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \nabla^2 \vec{A} = \mu_0 \vec{j}$$

We had some flexibility in choosing the scalar potential V because $\vec{E} = -\vec{\nabla} V$ is not changed if we add a constant to V, since $\vec{\nabla}$ (constant) $= 0$. Similarly here, if we add to $\vec{A}$ the gradient of a scalar function, $\vec{A}_2 = \vec{A}_1 + \vec{\nabla}\chi$, we have

$$\vec{B}_2 = \vec{\nabla} \times \vec{A}_2 = \vec{\nabla} \times \left(\vec{A}_1 + \vec{\nabla}\chi \right) = \vec{\nabla} \times \vec{A}_1 = \vec{B}_1$$

With this flexibility, we may choose $\vec{\nabla} \cdot \vec{A} = 0$. For suppose this is not true. Then

$$\vec{\nabla} \cdot \left(\vec{A}_1 + \vec{\nabla}\chi \right) = \vec{\nabla} \cdot \vec{A}_1 + \nabla^2 \chi = 0$$

So we have an equation for the function χ

$$\nabla^2 \chi = -\vec{\nabla} \cdot \vec{A}_1$$

Once we solve this we will have a vector $\vec{A}_2$ whose divergence is zero. Once we know that we can do this, we may just set $\vec{\nabla} \cdot \vec{A} = 0$ from the start. This is called the Coulomb gauge condition. With this choice, the equation for $\vec{A}$ is

$$\nabla^2 \vec{A} = -\mu_0 \vec{j} \tag{1}$$

We may look at this equation one component at a time (provided that we use Cartesian components.) Thus, for the $x-$component

$$\nabla^2 A_x = -\mu_0 j_x$$

This equation has the same form as the equation for V

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

and thus the solution will also have the same form:

$$A_x (\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{j_x (\vec{r}\,')}{R} d\tau'$$

and since we have an identical relation for each component, then

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}\,(\vec{r}')}{R}d\tau' \tag{2}$$

Now remember that $\vec{j}d\tau$ corresponds to $Id\vec{\ell}$, so if the current is confined in wires, the result is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{Id\vec{\ell}'}{R} \tag{3}$$

At this point we may stop and consider if there is any rule for magnetic field analagous to our RULE 1 for electric fields. Since there is no magetic charge, there is no "point charge" field. But we can use our expansion

$$\frac{1}{R} = \sum_{l=0}^{\infty}\frac{(r')^l}{r^{l+1}}P_l\left(\cos\theta'\right)$$

where $\vec{r}$ is on the polar axis. Then

$$\vec{A}\,(\vec{r} = r\hat{z}) = \frac{\mu_0}{4\pi}\sum \frac{I}{r^{l+1}}\int (r')^l\, P_l\left(\cos\theta'\right) d\vec{\ell}'$$

The $l = 0$ term is

$$\vec{A}_0 = \frac{\mu_0 I}{4\pi r}\int d\vec{\ell}'$$

Since the current flows in closed loops, $\int d\vec{\ell}' = 0$. (This result is actually more general, because in a static situation $\vec{\nabla}\cdot\vec{j} = 0$, and the lines of $\vec{j}$ also form closed loops.) This is the result we expected. The next term is

$$\vec{A}_1 = \frac{\mu_0 I}{4\pi r^2}\int r'\cos\theta' d\vec{\ell}' = \frac{\mu_0 I}{4\pi r^2}\int (\vec{r}'\cdot\hat{z})\, d\vec{\ell}'$$

We can use Stokes theorem to evaluate this integral.

$$\int \vec{u}\cdot d\vec{l} = \int\left(\vec{\nabla}\times\vec{u}\right)\cdot\hat{n}\; dA$$

Let $\vec{u} = \vec{c}\chi$ where $\vec{c}$ is a constant vector and χ is a scalar function. Then

$$\begin{aligned}\vec{c}\cdot\int \chi d\vec{l} &= \int\left(\vec{\nabla}\times\vec{c}\chi\right)\cdot\hat{n}dA \\ &= \int\left[\left(\vec{\nabla}\chi\right)\times\vec{c}\right]\cdot\hat{n}\; dA\end{aligned}$$

We may re-arrange the triple scalar product

$$\vec{c}\cdot\int \chi d\vec{l} = -\vec{c}\cdot\int\left(\vec{\nabla}\chi\right)\times\hat{n}\; dA$$

This is true for an arbitrary constant vector $\vec{c}$, so, with $\chi = (\vec{r}' \cdot \hat{z})$

$$\begin{aligned}
\int (\vec{r}' \cdot \hat{z})\, d\vec{l}' &= -\int \left[\vec{\nabla}' (\vec{r}' \cdot \hat{z})\right] \times \hat{n}' dA' \\
&= -\int \left[\hat{z} \times \left(\vec{\nabla}' \times \vec{r}'\right) + \left(\hat{z} \cdot \vec{\nabla}'\right) \vec{r}'\right] \times \hat{n}' dA' \\
&= -\int (0 + \hat{z} \times \hat{n}')\ dA' \\
&= -\hat{z} \times \int \hat{n}' dA' = \int \hat{n}' dA' \times \hat{z}
\end{aligned}$$

Note that $\hat{z}$ can come out of the integral because it is a constant. So

$$\vec{A}_1 = \frac{\mu_0 I}{4\pi r^2} \int \hat{n}' dA' \times \hat{z} = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{z} = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r}$$

where

$$\vec{m} = I \int \hat{n}' dA'$$

is the magnetic moment of the loop. The corresponding magnetic field is

$$\begin{aligned}
\vec{B}_1 &= \vec{\nabla} \times \left[\frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r}\right] \\
&= \frac{\mu_0}{4\pi} \left(-\frac{3}{r^4} \hat{r} \times (\vec{m} \times \vec{r}) + \frac{1}{r^3} \vec{\nabla} \times (\vec{m} \times \vec{r})\right) \\
&= \frac{\mu_0}{4\pi} \left(-\frac{3}{r^3} \left[\vec{m} - \hat{r} (\vec{m} \cdot \hat{r})\right] + \frac{1}{r^3} \left(-\left(\vec{m} \cdot \vec{\nabla}\right) \vec{r} + \vec{m} \left(\vec{\nabla} \cdot \vec{r}\right)\right)\right) \\
&= \frac{\mu_0}{4\pi r^3} (-3\vec{m} + 3\vec{r} (\vec{m} \cdot \hat{r}) - \vec{m} + 3\vec{m}) \\
&= \frac{\mu_0}{4\pi r^3} \left[3\vec{r} (\vec{m} \cdot \hat{r}) - \vec{m}\right]
\end{aligned}$$

This is a dipole field. Thus the magnetic equivalent of RULE 1 is :

> At a great distance from a current distribution, the magnetic field is a dipole field

Here is another useful result:

$$\oint_C \vec{A} \cdot d\vec{\ell} = \int_S \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\ da = \int_S \vec{B} \cdot \hat{n}\ da = \Phi_B \qquad (4)$$

Thus the circulation of $\vec{A}$ around a curve C equals the magnetic flux through any surface S spanning the curve.

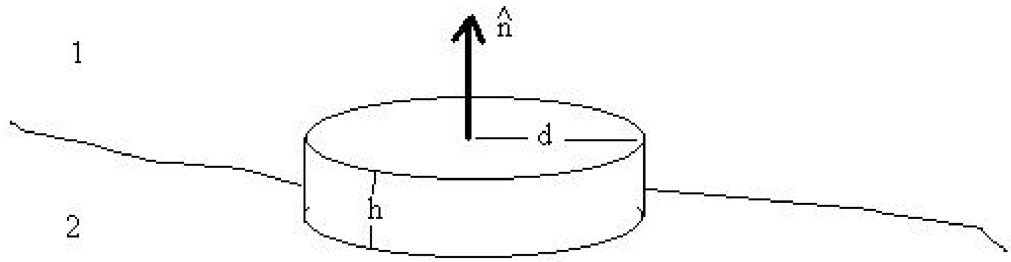
Boundary conditions for $\vec{B}$

We start with the Maxwell equations. Remember, if the equation has a divergence we integrate over a small volume (pillbox) that crosses the boundary.

But if the equation has a curl, we integrate over a rectangular surface that lies perpendicular to the surface.

So we start with $\vec{\nabla} \cdot \vec{B} = 0$

$$\int \vec{\nabla} \cdot \vec{B} d\tau = 0 = \oint_S \vec{B} \cdot d\vec{A}$$

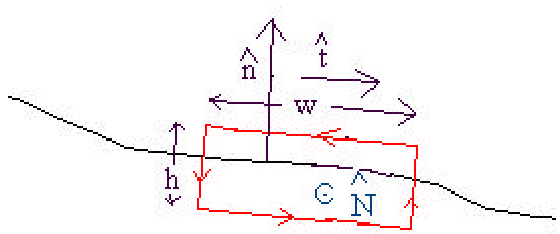


But because we chose $h \ll d$, the integral over the sides is negligible, and on the bottom side $d\vec{A}_2 = -\hat{n} dA$, so we have

$$\left(\vec{B}_1 - \vec{B}_2\right) \cdot \hat{n} = 0 \tag{5}$$

The normal component of $\vec{B}$ is continuous.

For the curl equation, we use the rectangle shown:



Then

$$\begin{aligned} \int_S \left(\vec{\nabla} \times \vec{B}\right) \cdot d\vec{A} &= \int_S \mu_0 \vec{j} \cdot d\vec{A} \\ \oint_C \vec{B} \cdot d\vec{\ell} &= \mu_0 \int \vec{j} \cdot \hat{N} dh\, w \\ \left(\vec{B}_1 - \vec{B}_2\right) \cdot \left(-\hat{t}\right) w &= \mu_0 \mu_0 \left(\int \vec{j} dh\right) \cdot \hat{N}\, w \\ \left(\vec{B}_1 - \vec{B}_2\right) \cdot \left(-\hat{n} \times \hat{N}\right) &= \mu_0 \vec{K} \cdot \hat{N} \end{aligned}$$

Rearrange the triple scalar product on the left to get

$$-\left[\left(\vec{B}_1 - \vec{B}_2\right) \times \hat{n}\right] \cdot \hat{N} = \mu_0 \vec{K} \cdot \hat{N}$$

Since we may orient the rectangle so that $\hat{N}$ is any vector in the surface, we have

$$\hat{n} \times \left(\vec{B}_1 - \vec{B}_2\right) = \mu_0 \vec{K} \tag{6}$$

Thus the tangential component of $\vec{B}$ has a discontinuity that depends on the surface current density $\vec{K}$. Crossing both sides with $\hat{n}$, we get an alternate version:

$$\begin{aligned} \left[\hat{n} \times \left(\vec{B}_1 - \vec{B}_2\right)\right] \times \hat{n} &= \mu_0 \vec{K} \times \hat{n} \\ \left(\vec{B}_1 - \vec{B}_2\right) - \hat{n}\left[\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right)\right] &= \mu_0 \vec{K} \times \hat{n} \end{aligned}$$

But now we may make use of (5) to obtain

$$\left(\vec{B}_1 - \vec{B}_2\right) = \mu_0 \vec{K} \times \hat{n} \tag{7}$$

What about the vector potential? Remember that for the scalar potential V we were able to show that V is continuous across the surface (in most cases). When we find $\vec{A}$ we first choose a gauge condition. The Coulomb gauge condition is

$$\vec{\nabla} \cdot \vec{A} = 0$$

and then we can use our usual pillbox trick to show that

$$\vec{A} \cdot \hat{n} \quad \text{is continuous} \tag{8}$$

For the tangential component, we make use of equation (4). Then, using the rectangle,

$$\begin{aligned} \oint_C \vec{A} \cdot d\vec{\ell} &= \Phi_B = \vec{B} \cdot \hat{N} wh \\ \left(\vec{A}_1 - \vec{A}_2\right) \cdot \left(-\hat{t}\right) w &= \vec{B} \cdot \hat{N} wh \to 0 \quad \text{as } h \to 0 \end{aligned}$$

Thus we have

$$\vec{A} \cdot \hat{t} \quad \text{is continuous} \tag{9}$$

These two result taken together show that the vector potential as a whole is also continuous across the boundary.

Finally let's put $\vec{A}$ into equation (6):

$$\vec{\nabla} \times \left(\vec{A}_1 - \vec{A}_2\right) = \mu_0 \vec{K} \times \hat{n}$$

So the derivatives of $\vec{A}$ have a discontinuity. But which ones? Let's expand

$$\hat{n} \times \vec{B} = \hat{n} \times \left(\vec{\nabla} \times \vec{A}\right) = n_i \vec{\nabla} A_i - \left(\hat{n} \cdot \vec{\nabla}\right) \vec{A}$$

Then

$$\hat{n} \times \left(\vec{B}_1 - \vec{B}_2\right) = n_i \vec{\nabla} \left(A_{i,1} - A_{i,2}\right) - \left(\hat{n} \cdot \vec{\nabla}\right) \left(\vec{A}_1 - \vec{A}_2\right) = \mu_0 \vec{K} \qquad (10)$$

But we have shown that each component of $\vec{A}$ is continuous at the surface. So the components of

$$\vec{\nabla} \left(A_{i,1} - A_{i,2}\right)$$

parallel to the surface must be zero. Thus only the normal derivatives remain. Then the normal component of equation (10) is identically zero, and the only non-zero components of the boundary condition are the tangential components

$$\left(\hat{n} \cdot \vec{\nabla}\right) \left(\vec{A}_1 - \vec{A}_2\right)_{\tan} = -\mu_0 \vec{K} \qquad (11)$$

Now this is neat. Each component of $\vec{A}$ satisfies Laplace's equation with Neumann boundary conditions, and so it must have a unique solution, as we already proved for V.

Magnetic scalar potential

When we have the special case of $\vec{j} \equiv 0$, $\vec{\nabla} \times \vec{B} = 0$ and we may use a magnetic scalar potenial Φ_{mag}. This can be useful if the current is confined to lines or sheets, because we can create a nice boundary-value problem for Φ_{mag}.

$$\vec{B} = -\vec{\nabla} \Phi_{\text{mag}}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \nabla^2 \Phi_{\text{mag}} = 0 \qquad (12)$$

$$B_{\text{normal}} \text{ continuous} \Rightarrow \hat{n} \cdot \vec{\nabla} \Phi_{\text{mag}} \text{ is continuous} \qquad (13)$$

$$\hat{n} \times \left(\vec{B}_1 - \vec{B}_2\right) = \mu_0 \vec{K} \Rightarrow \hat{n} \times \vec{\nabla} \left(\Phi_{\text{mag}\,1} - \Phi_{\text{mag}\,2}\right) = -\mu_0 \vec{K} \qquad (14)$$

Let's use these boundary conditions to find the potential due to a spinning spherical shell of charge. The current is confined to the surface and has the value

$$\vec{K} = \sigma \vec{v} = \sigma \vec{\omega} \times \vec{r} = \sigma \omega a \sin\theta \hat{\phi}$$

where in the last expression put the $z-$axis along the rotation axis. We will take σ to be a constant. The equation for Φ_{mag} in the region entirely inside (or entirely outside) the sphere is (1 with $\vec{j} = 0$)

$$\nabla^2 \Phi_{\text{mag}} = 0$$

and because we have azimuthal symmetry, the solution is of the form

$$\begin{aligned} \Phi_{\text{in}} &= \sum_{l=1}^{\infty} C_l r^l P_l(\cos\theta) \\ \Phi_{\text{out}} &= \sum_{l=1}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos\theta) \end{aligned}$$

We have omitted the $l = 0$ term because it contributes zero field inside, and we know there can be no monopole term outside. What else do we know? At the boundary, from (13)

$$\begin{aligned}
\left.\frac{\partial \Phi_{\mathrm{mag,out}}}{\partial r}\right|_{r=a} - \left.\frac{\partial \Phi_{\mathrm{mag,in}}}{\partial r}\right|_{r=a} &= 0 \\
\sum_{l=1}^{\infty} l C_l a^{l-1} P_l(\cos\theta) &= -\sum_{l=1}^{\infty} (l+1) \frac{D_l}{a^{l+2}} P_l(\cos\theta) \\
C_l &= -\frac{D_l}{a^{2l+1}} \frac{l+1}{l} \quad l > 0 \qquad (15)
\end{aligned}$$

and from (14).

$$\begin{aligned}
\frac{1}{a}\left(\left.\frac{\partial \Phi_{\mathrm{mag,out}}}{\partial \theta}\right|_{r=a} - \left.\frac{\partial \Phi_{\mathrm{mag,in}}}{\partial \theta}\right|_{r=a} \right) &= -\mu_0 \sigma a \omega \sin\theta \\
\frac{1}{a\sin\theta}\left(\left.\frac{\partial \Phi_{\mathrm{mag,out}}}{\partial \phi}\right|_{r=a} - \left.\frac{\partial \Phi_{\mathrm{mag,in}}}{\partial \phi}\right|_{r=a} \right) &= 0
\end{aligned}$$

The last equation is automatically satisfied. Thus the final condition we need to satisfy is

$$\sum_{l=1}^{\infty} \frac{D_l}{a^{l+2}} \frac{\partial}{\partial \theta} P_l(\cos\theta) - \sum_{l=1}^{\infty} C_l a^{l-1} \frac{\partial}{\partial \theta} P_l(\cos\theta) = -\mu_0 \sigma a \omega \sin\theta$$

Now since $P_1(\cos\theta) = \cos\theta$ and $\frac{\partial}{\partial\theta}\cos\theta = -\sin\theta$, the first term in the sum is

$$-\left(\frac{D_1}{a^3} - C_1 \right) \sin\theta$$

so we may satisfy the boundary conditons by taking

$$\frac{D_1}{a^3} - C_1 = \mu_0 \sigma a \omega$$

and all the other C_l, $D_l = 0$. Then equation (15) gives

$$\frac{D_1}{a^3} + \frac{D_1}{a^3}\frac{2}{1} = \mu_0 \sigma a \omega \Rightarrow D_1 = \frac{\mu_0 \sigma a^4 \omega}{3}$$

and then

$$C_1 = -2\frac{\mu_0 \sigma a \omega}{3}$$

So

$$\Phi_{\mathrm{mag}} = \left\{ \begin{array}{cc} -\frac{2}{3}\mu_0 \sigma a \omega r \cos\theta & \text{inside} \\ \frac{1}{3}\mu_0 \sigma a^2 \omega \frac{a^2}{r^2} \cos\theta & \text{outside} \end{array} \right\}$$

giving a field

$$\vec{B} = \left\{ \begin{array}{cc} \frac{2}{3}\mu_0 \sigma a \omega \hat{z} & \text{inside} \\ \mu_0 \sigma a \omega \frac{a^3}{3r^3}\left(2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta} \right) & \text{outside} \end{array} \right\}$$

Thus the field inside is uniform and the field outside is a pure dipole field. The dipole moment is

$$m = \frac{4\pi}{3}\sigma a^4 \omega$$

The dimensions of m are

$$\frac{\text{charge}}{\text{area}} \frac{(\text{length})^4}{\text{time}} = \frac{\text{charge}}{\text{time}} \times (\text{length})^2 = \text{current} \times \text{area}$$

which is correct. You should verify that you get the same m by summing current loops.

Compare this solution with Griffiths' example 5.11. Which method do you think is easier?