**The Biot-Savart Law**

This semester we want to look at fields that are constant in time. That means that we should not attempt to consider the magnetic field due to a single charge, because only moving charges produce magnetic fields (just as only moving charges experience magnetic force). The simplest source of magnetic field is *steady current*, current that does not change in time, and the closest we can come to a point source is a differential piece. The magnetic field produced by a wire segment of length $d\ell$ carrying current $I$ at a point $P$ is

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

This law is actually a lot like Coulomb's law. It's an inverse square law, and it depends on the vector $\vec{r}$ that points from the wire to $P$. The new complication is that the source is a vector, and so the Biot-Savart law involves the cross product. That means that the magnetic field at $P$ is perpendicular to both $d\vec{\ell}$ and $\hat{r}$. The field produced by a long straight wire forms circles centered on the wire.

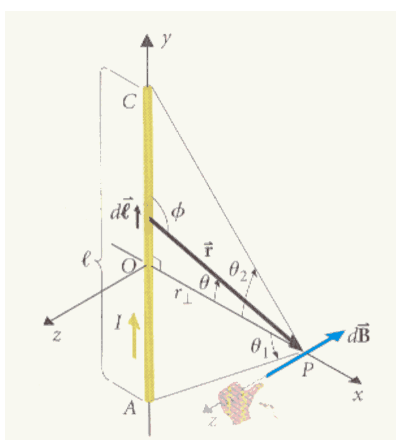
To get the total $\vec{B}$ we have to integrate over the whole source:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int I \frac{d\vec{\ell} \times \hat{R}}{R^2}$$

where now we describe the source point by the vector $\vec{r}'$, as we did for electric fields, and the vector $\vec{R}$ points from the source to $P$ :

$$\vec{R} = \vec{r} - \vec{r}'$$

Let's begin by finding the magnetic field due to a straight wire segment.




We put the $y-$axis along the wire in the direction of $I$, and the $x-$axis through $P$, which is distance $s$ from the wire. Model the wire as a collection of differential elements $d\vec{\ell} = d\ell\hat{y}$ at position $y'$. Then

$$\vec{R} = s\hat{x} - y'\hat{y}$$

and

$$\begin{aligned}\vec{B}(P) &= \frac{\mu_0}{4\pi}\int_{y_1}^{y_2} I\frac{dy'\hat{y}\times(s\hat{x}-y'\hat{y})}{\left[s^2+(y')^2\right]^{3/2}} \\ &= \frac{\mu_0}{4\pi}\int_{y_1}^{y_2} I\frac{dy'\,s\,(-\hat{z})}{\left[s^2+(y')^2\right]^{3/2}}\end{aligned}$$

We integrate from one end of the wire to the other. Now $I$, $s$, and the unit vector $\hat{z}$ are independent of $y'$, so we can take them out of the integral:

$$\vec{B}(P) = -\frac{\mu_0 I s}{4\pi}\hat{z}\int \frac{dy'}{\left[s^2+(y')^2\right]^{3/2}}$$

To do the integral, we use a tangent substitution. Let $\tan\theta = y'/s$. Then $\sec^2\theta$ $d\theta = dy'/s$.

$$\begin{aligned}\vec{B}(P) &= -\frac{\mu_0 I s}{4\pi}\hat{z}\int_{\theta_1}^{\theta_2}\frac{s\sec^2\theta d\theta}{\left[s^2+s^2\tan^2\theta\right]^{3/2}} \\ &= -\frac{\mu_0 I s^2}{4\pi}\hat{z}\int_{\theta_1}^{\theta_2}\frac{\sec^2\theta d\theta}{s^3\left[\sec^2\theta\right]^{3/2}} \\ &= -\frac{\mu_0 I}{4\pi s}\hat{z}\int_{\theta_1}^{\theta_2}\cos\theta d\theta \\ &= -\frac{\mu_0 I}{4\pi s}\hat{z}\left(\sin\theta_2-\sin\theta_1\right)\end{aligned}$$

The angles $\theta_1$ and $\theta_2$ are shown in the diagram.

We have to be aware of several important facts if we want to use this expression correctly. First, we chose our $y$ axis so that $y$ increases in the direction of $I$. That means that $y_2$ is always bigger than $y_1$, and also $\theta_2 > \theta_1$ as a result. However, there are several possibilities:

1. $y_1$ and $y_2$ (and hence $\theta_1$ and $\theta_2$) may both be positive
2. $y_1 > 0$ but $y_2 < 0$ (and hence $\theta_1 > 0$ and $\theta_2 < 0$)
3. $y_1$ and $y_2$ (and hence $\theta_1$ and $\theta_2$) may both be negative

An infinitely long wire would be in case 2.

$$\tan\theta_1 = \frac{y_{\text{top end}}}{s} \to \infty \text{ as } y_{\text{top end}} \to \infty$$

and thus

$$\theta_1 = \frac{\pi}{2}$$

while

$$\tan\theta_2 = \frac{y_{\text{bottom end}}}{s} \to -\infty \text{ as } y_{\text{bottom end}} \to -\infty$$

so

$$\theta_2 = -\frac{\pi}{2}$$

and so

$$\vec{B}_{\text{long wire}} = -\frac{\mu_0 I}{4\pi s}\hat{z}\left[1-(-1)\right] = -\frac{\mu_0 I}{2\pi s}\hat{z}$$

Check the direction and convince yourself it is correct. Note that $\vec{B}$ is independent of $y$,and $z$ as it must be due to the translational and rotational symmetry of the system. Also note that the field decreases as 1/distance from the wire. This is exactly the same dependence that we found for the electric field due to an infinite line charge.

*Aside.* Now we are ready to look at an important system that is used to define the current unit ampere. We have two infinitely long wires, each carrying current $I$, and separated by distance $s$ The one on the left produces a magnetic field $-\frac{\mu_0 I}{2\pi s}\hat{z}$ at the position of the one on the right. Thus the second wire experiences a force

$$d\vec{F} = I d\ell\hat{y} \times \left(-\frac{\mu_0 I}{2\pi s}\hat{z}\right) = -\frac{\mu_0 I^2}{2\pi s}\hat{x}\ d\ell$$

on each segment $d\ell$. Thus there is a force per unit length

$$\frac{d\vec{F}}{d\ell} = -\frac{\mu_0 I^2}{2\pi s}\hat{x}$$

The second wire is attracted toward the first. Go through the analysis to convince yourself that there is an equal and opposite force per unit length on the wire on the left. The ampere is defined by measuring the force and adjusting the current until the force/length is $2\times 10^{-7}$ N/m when the wire separation is $s = 1$ m. The resulting current is 1 A by definition.

Now Griffiths claims that there is no electric force because the wires are electrically neutral, but that's wrong. In a normal lab situation, the system must have some resistance, so let's look at an idealized system with the two long wires being superconductors, but the connectors at infinity having a resistance $R$. Then we need a potential difference $\Delta V = IR$ between the two wires. This can be accomplished if each carries a uniform line charge density $\pm\lambda$. With origin on the positively-charged wire and $x-$axis pointing toward the negatively-charged wire, the electric field at a point between the wires is

$$\vec{E} = \frac{\lambda\hat{x}}{2\pi\varepsilon_0}\left(\frac{1}{x} + \frac{1}{s-x}\right)$$

The potential difference is then

$$\begin{aligned}\Delta V &= \int \vec{E}\cdot d\vec{l} = \frac{\lambda}{2\pi\varepsilon_0}\int_a^{s-a}\left(\frac{1}{x}+\frac{1}{s-x}\right)dx = \frac{\lambda}{2\pi\varepsilon_0}\left[\ln x - \ln(s-x)\right] \\ IR &= \frac{\lambda}{2\pi\varepsilon_0}\left(2\ln\frac{s-a}{a}\right)\end{aligned}$$

where $a$ is the radius of each wire. Thus

$$\lambda = \frac{\pi\varepsilon_0 IR}{\ln(s/a-1)}$$

The charge density $\lambda \to 0$ as $a \to 0$, but is not zero for a real wire. Then the electric force on a segment of wire is

$$\begin{aligned} dF &= E\lambda dl = \frac{\lambda^2}{2\pi\varepsilon_0 s} dl = \left[\frac{\pi\varepsilon_0 IR}{\ln(s/a-1)}\right]^2 \frac{dl}{2\pi\varepsilon_0 s} \\ \frac{dF}{dl} &= \frac{I^2}{2\pi s}\frac{\pi^2\varepsilon_0 R^2}{[\ln(s/a-1)]^2} \end{aligned}$$

So

$$\frac{F_{\text{elec}}}{F_{\text{mag}}} = \frac{\pi^2\varepsilon_0 R^2}{\mu_0 [\ln(s/a-1)]^2}$$

Let's put in some numbers. Let $R = 1\ \Omega$, $s = 10$ cm, $a = 1$ mm. Then we get

$$\begin{aligned} \frac{F_{\text{elec}}}{F_{\text{mag}}} &= \pi^2 \frac{(1\ \Omega)^2 \left(8.85 \times 10^{-12}\ \text{F/m}\right)}{\left(4\pi \times 10^{-7}\ \text{N/A}^2\right)[\ln 100 - 1]^2} \\ &= \frac{\pi \times 8.85 \times 10^{-12}}{(4 \times 10^{-7}\ )(4.605\,2-1)^2} \frac{\Omega^2\text{F/m}}{\text{N/A}^2} \\ &= 5.3 \times 10^{-6} \frac{\Omega^2\text{F/m}}{\text{N/A}^2} \end{aligned}$$

Let's check the units.

$$\Omega = \frac{\text{V}}{\text{A}}$$

and

$$\text{F}=\frac{\text{C}}{\text{V}}$$

so

$$\frac{\Omega^2\text{F/m}}{\text{N/A}^2} = \frac{\text{V}^2}{\text{N}}\frac{\text{C}}{\text{V}\cdot\text{m}} = \frac{\text{V}\cdot\text{C}}{\text{N}\cdot\text{m}} = \frac{\text{J}}{\text{J}}$$

Thus the ratio is dimensionless, as required.

The ratio is small, but is not zero! A resistance of k$\Omega$ instead of $\Omega$ would make the ratio $> 1$.

**Ampere's Law**

We have shown above that a long wire produces a magnetic field that curls around the wire. If we choose a circular path that follows the field line, we have

$$\oint_C \vec{B} \cdot d\vec{l} = \int_0^{2\pi} \frac{\mu_0 I}{2\pi s} s d\theta = \mu_0 I$$

The result is independent of $s$. Now let's generalize the result to a curve of arbitrary shape. On such a curve

$$d\vec{l} = ds\ \hat{s} + sd\theta\ \hat{\theta} + dz\ \hat{z}$$

and

$$\vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi s}\hat{\theta} \cdot \left(ds\ \hat{s} + sd\theta\ \hat{\theta} + dz\ \hat{z}\right) = \frac{\mu_0 I}{2\pi} d\theta$$

Thus

$$\oint_C \vec{B} \cdot d\vec{l} = \oint_C \frac{\mu_0 I}{2\pi} d\theta$$

If the curve surrounds the wire, then the limits are 0 to $2\pi$, and we get $\mu_0 I$, as before. But if the wire is outside the curve, the limits go from some minimum value $\theta_1$ to a maximum $\theta_2$ and back to $\theta_1$, giving zero as the result of the integral. Now we can add as many wires as we like, to get the general rule called Ampere's Law:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \,(\text{total current through } C)$$

Now let's use the fact that $I$ is the flux of $\vec{j}$.

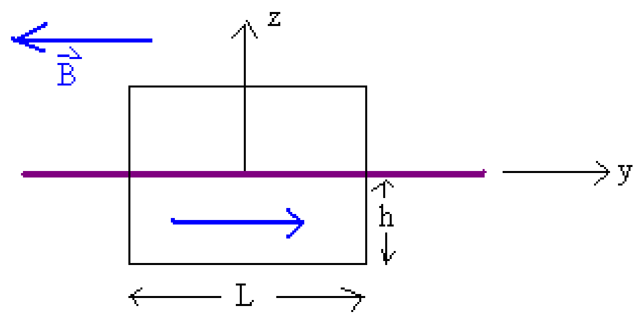
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot \hat{n}\ dA \tag{1}$$

which is the integral form of Ampere's law. It is true generally for any $\vec{j}$, not only currents in straight wires. As with Gauss' law, this global statement is useful as a tool for finding $\vec{B}$ only in cases with sufficient symmetry: plane, spherical or cylindrical symmetry, but also toroids (See G Ex 5.10). See LB page 910 for the method.

Suppose we have current confined to a thin sheet of conductor in the $x-y$ plane. The current is described by a surface current $K$ (units A/m).

$$\vec{K} = \int \vec{j}\ dz = K\hat{x}$$

If we model the plane as a collection of wires, each of thickness $dy$ and carrying current $Kdy$, we can see that the magnetic field above the plane at $z > 0$ will be the superposition of fields circulating around the wires. This superposition gives a field in the $-y-$direction, and parallel to the sheet. Below the sheet, the field is reversed. We choose an Amperian curve that is a rectangle of length $L$ and width $2h$, lying parallel to the $x-z$ plane. By placing it symmetrically about the sheet we guarantee that the magnitude of $B$ on the top side equals the magnitude on the bottom side.




Then integrating around this curve, we have

$$\oint_C \vec{B} \cdot d\vec{l} = 2BL = \mu_0 \int \vec{j} \cdot \hat{n}\ dA = \mu_0 KL$$

Thus

$$B = \mu_0 \frac{K}{2}$$

The result is independent of $L$, as it must be, but it is also independent of $h$. We saw a similar result for the electric field due to a charged sheet. Finally

$$\begin{aligned} \vec{B} &= -\mu_0 \frac{K}{2} \hat{y} \text{ above the sheet} \\ &= +\mu_0 \frac{K}{2} \hat{y} \text{ below the sheet} \end{aligned}$$

This result is important in understanding how the Earth'smagnetic field can shield us from charged particles arriving from the sun. Charged particles spiral around magnetic field lines. When paricles from the sun reach the Earth's field, the electrons spiral one way and the ions the other. Both signs of charge create a current sheet of the same sign, and that sheet acts to reduce the field outside the sheet and double it inside. The net result is that the Earth;s field is cut off at the magnetopause. This boundary prevents particles from plunging directly to the Earth's surface.

Finally we can use Ampere's law to get a local result. We use Stokes' theorem to modify the LHS

$$\int_S \left(\vec{\nabla} \times \vec{B}\right) \cdot \hat{n}\ dA = \mu_0 \int_S \vec{j} \cdot \hat{n}\ dA$$

Now this is true for absolutely any curve $C$ and any surface $S$ spanning the curve, so

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \tag{2}$$

As with the curl equation for $\vec{E}$, this result is true only for static fields.

We asserted earlier that $\vec{\nabla} \cdot \vec{B} = 0$ because there are no magnetic charges. Now let's verify this starting with the Biot-Savart law.First we re-write the B-S law in terms of $\vec{j}$. A general current distribution may be written as a collection of current loops, each having cross sectional area $dA'$.

$$\begin{aligned}
\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} \sum_{\text{all loops}} \int I \frac{d\vec{\ell} \times \hat{R}}{R^2} = \frac{\mu_0}{4\pi} \sum_{\text{all loops}} \int j dA' \frac{d\vec{\ell} \times \hat{R}}{R^2} \\
&= \frac{\mu_0}{4\pi} \sum_{\text{all loops}} \int dA' d\ell \frac{\vec{j} \times \hat{R}}{R^2} \\
\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} \int_V \frac{\vec{j}(\vec{r}') \times \hat{R}}{R^2} d\tau' \qquad (3)
\end{aligned}$$

Then

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \vec{\nabla} \cdot \frac{\mu_0}{4\pi} \int_V \frac{\vec{j}(\vec{r}') \times \hat{R}}{R^2} d\tau'$$

Now the grad operator differentiates the unprimed coordinates, so we may move it inside the integral.

$$\begin{aligned}
\vec{\nabla} \cdot \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} \int_V \vec{\nabla} \cdot \left( \frac{\vec{j}(\vec{r}') \times \hat{R}}{R^2} \right) d\tau' \\
&= \frac{\mu_0}{4\pi} \int_V \left[ \vec{j}(\vec{r}') \cdot \left( \vec{\nabla} \times \frac{\hat{R}}{R^2} \right) - \frac{\hat{R}}{R^2} \cdot \vec{\nabla} \times \vec{j}(\vec{r}') \right] d\tau'
\end{aligned}$$

But $\vec{\nabla} \times \vec{j}(\vec{r}') \equiv 0$, so

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \left[ \vec{j}(\vec{r}') \cdot \left( \vec{\nabla} \times \frac{\hat{R}}{R^2} \right) \right] d\tau'$$

Now

$$\frac{\hat{R}}{R^2} = -\vec{\nabla} \frac{1}{R}$$

and the curl of any gradient is zero, so $\vec{\nabla} \cdot \vec{B} = 0$, as required.

**Maxwell's equations for static fields**

We now have all four Maxwell equations, limited to the case of static fields. They are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}; \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0; \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

Here we have grouped them according to the differential operation that appears (divergence or curl). But we could also group them this way:

$$\vec{\nabla} \times \vec{E} = 0; \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}; \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

where the top two are homogeneous equations (RHS=0) and the bottom two display the sources $\rho$ and $\vec{j}$ of the fields. To these we add the force law:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

and charge conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

These equations illustrate the similarities and differences between $\vec{E}$ and $\vec{B}$. The two source-free equations indicate the lack of magnetic charge– there are (essentially) no magnetic monopoles. The fact that $\vec{E}$ has a non-zero divergence means that $\vec{E}$ field lines radiate outward from positive charge and into negative charge. Because $\vec{B}$ has a non-zero curl, the field lines wrap around the current sources. Since $\vec{\nabla} \cdot \vec{B} = 0$, magnetic field lines do not begin or end– they form closed loops (which may extend to infinity before they close).

A fact that does not emerge obviously from these equations is that magnetic forces are usually much weaker than electric forces, unless we contrive situations that make the electric force zero, or almost zero, as is usually the case with currents in wires, or permanent magnets. This is much more apparent in the cgs Gaussian unit system, in which the force law is

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

In this unit system $\vec{E}$ and $\vec{B}$ have the same units, and we can see that for comparable fields we'd need relativistic speeds ($v \sim c$) to make the forces comparable. (See LB pg 914 for an example.)